Total Restrained Domination in Trees

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Abstract

Let $G = (V, E)$ be a graph. A set $S \subseteq V$ is a total restrained dominating set if every vertex is adjacent to a vertex in S and every vertex of $V - S$ is adjacent to a vertex in $V - S$. The total restrained domination number of G, denoted by $\gamma_{tr}(G)$, is the smallest cardinality of a total restrained dominating set of G. We show that if T is a tree of order n, then $\gamma_{tr}(T) \geq \lceil \frac{n+2}{2} \rceil$. Moreover, we show that if T is a tree of order $n \equiv 0$ mod 4, then $\gamma_{tr}(T) \geq \lceil \frac{n+2}{2} \rceil + 1$. We then constructively characterize the extremal trees T of order n achieving these lower bounds.

1 Introduction

In this paper, we follow the notation of [1]. Specifically, let $G = (V, E)$ be a graph with vertex set V and edge set E. Moreover, the notation P_n will denote the path of order n. A set $S \subseteq V$ is a *dominating set* of G if every vertex not in S is adjacent to a vertex in S. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set. The concept of domination in graphs, with its many variations, is now well studied in graph theory. The recent book of Chartrand and Lesniak [1] includes a chapter on domination. A thorough study of domination appears in [3, 4].

In this paper, we continue the study of a variation of the domination theme, namely that of total restrained domination. A set $S \subseteq V$ is a *total restrained dominating set* (denoted **TRDS**) if every vertex is adjacent to a vertex in S and every vertex in $V - S$ is also adjacent to a vertex in $V - S$. Every graph has a total restrained dominating set, since $S = V$ is such a set. The *total restrained domination number* of G, denoted by $\gamma_{tr}(G)$, is the minimum cardinality of a **TRDS** of G. A **TRDS** of cardinality $\gamma_{tr}(G)$ will be called a $\gamma_{tr}(G)$*-set.*

The concept of total restrained domination was introduced by Chen, Ma and Sun in [2], and further studied by Zelinka in [6]. We may note that the concept of total restrained domination was also introduced by Telle and Proskurowski [5], albeit indirectly, as a vertex partitioning problem. Here conditions are imposed on a set S, the complementary set $V-S$ and on edges between the sets S and $V-S$. For example, if we require that every vertex in $V-S$ should be adjacent to some other vertex of $V-S$ (the condition on the set $V-S$) and to some vertex in S (the condition on edges between the sets S and $V-S$), and every vertex in S is also adjacent to some vertex in S (the condition on edges among vertices of S), then S is a **TRDS**.

We refer to a vertex of degree 1 in a tree T as a *leaf* of T. A vertex adjacent to a leaf we call a *remote vertex* of T. For $v \in V(T)$ and a leaf ℓ of T, the path $vx_1 \dots x_k\ell$ is called a $v-L$ *path* if $\deg x_i = 2$ for each i. If the vertex v need not be specified, a $v-L$ path is also called an *endpath*.

We show that if T is a tree of order n, then $\gamma_{tr}(T) \geq \lceil \frac{n+2}{2} \rceil$. Moreover, we constructively characterize the extremal trees T of order n achieving this lower bound. Lastly, we show that if T is a tree of order $n \equiv 0 \bmod 4$, then $\gamma_{tr}(T) \geq \lceil \frac{n+2}{2} \rceil + 1$, and also constructively characterize the extremal trees T of order n achieving this lower bound.

2 The lower bound

The following result was established in [2], using a more cumbersome proof. As we shall see, this result will be useful in establishing a sharp lower bound on the total restrained domination number of a tree.

Proposition 1 *If $n \geq 2$ is an integer, then $\gamma_{tr}(P_n) = n - 2\lfloor \frac{n-2}{4} \rfloor$.*

Proof. Suppose S is a **TRDS** of P_n, whose vertex set is $V = \{v_1, \dots, v_n\}$. Note that $v_1, v_2 \in S$. Moreover, any component of $V-S$ is of size exactly two. Each component is adjacent to a vertex of S, which, in turn, is adjacent to another vertex of S. Suppose there are m such components. Then $2m + 2m + 2 \leq n$ and so $m \leq \lfloor \frac{n-2}{4} \rfloor$. Thus $|S| = n - 2m \geq n - 2\lfloor \frac{n-2}{4} \rfloor$. On the other hand, $V - \{v_i \mid i \in \{3, 4, 7, 8, \dots, 4\lfloor \frac{n-2}{4} \rfloor - 1, 4\lfloor \frac{n-2}{4} \rfloor\}\}$ is a **TRDS** of P_n, whence $\gamma_{tr}(P_n) = n - 2\lfloor \frac{n-2}{4} \rfloor$. $\square$

Corollary 2 *If $n \geq 2$ is an integer, then $\gamma_{tr}(P_n) \geq \lceil \frac{n+2}{2} \rceil$.*

Proof. Since $n - 2\lfloor \frac{n-2}{4} \rfloor \geq \lceil \frac{n+2}{2} \rceil$, the result follows from Proposition 1. $\square$

Let $T = (V, E)$ be a tree and $v, a, b \in V$ such that $\deg v \geq 3$ and $a, b \in N(v)$. Let ℓ_b be a leaf of the component of $T - v$ that contains b. Then the tree T' which arises from T by deleting the edge va and joining a to ℓ_b is called a (v, a, b)*-pruning* of T.

Theorem 3 *If T is a tree of order $n \geq 2$, then $\gamma_{tr}(T) \geq \lceil \frac{n+2}{2} \rceil$.*

Proof. We use induction on n. It is easy to check that the result is true for all trees T of order $n \leq 8$. Suppose, therefore, that the result is true for all trees of order less than n, where $n \geq 9$. Let $\gamma_{tr} = \min\{\gamma_{tr}(T) \mid T \text{ is a tree of order } n\}$. We will show that $\gamma_{tr} \geq \lceil \frac{n+2}{2} \rceil$.

Let $\mathcal{T} = \{T \mid T \text{ is a tree of order } n \text{ such that } \gamma_{tr}(T) = \gamma_{tr}\}$. Among all trees in $\mathcal{T}$, let T be chosen so that the sum $s(T)$ of the degrees of its vertices of degree at least 3 is minimum. If $s(T) = 0$, then $T \cong P_n$, and so $\gamma_{tr} = \gamma_{tr}(P_n) \geq \lceil \frac{n+2}{2} \rceil$. Suppose, therefore, that $s(T) \geq 1$. Since $s(T) \geq 1$, there exists a vertex v such that $\deg(v) \geq 3$. Let S be a $\gamma_{tr}(T)$-set of T.

Claim 1 *If v is a vertex of degree at least 3, then*
(i) $v \notin S$,
(ii) v is adjacent to exactly one vertex of S,
(iii) $\deg(v) = 3$.

Proof. Suppose $v \in S$. Then there exist $a, b \in N(v)$ such that $b \in S$. Let T' be a (v, a, b)-pruning of T. Then S is a **TRDS** of T', and so, by definition of γ_{tr}, we have that $\gamma_{tr} \leq \gamma_{tr}(T') \leq |S| = \gamma_{tr}$. Hence, $T' \in \mathcal{T}$. However, as $s(T') < s(T)$, we obtain a contradiction.

Thus, assume $v \notin S$ and let $a, b \in N(v)$ such that $a \notin S$ and $b \in S$. If $c \in N(v) - \{a, b\}$ is in S, then, by considering the (v, b, c)-pruning of T, we obtain a contradiction as before. We therefore assume that b is the only vertex in S which is adjacent to v.

Suppose $\deg(v) \geq 4$, let $\{c_1, \ldots, c_{\deg(v)-2}\} = N(v) - \{a, b\}$, let $c = c_1$ and let ℓ_b be a leaf of the component of $T - v$ that contains b. Let T' be the tree which arises from T by deleting the edges vc_i for $i = 1, \ldots, \deg(v) - 2$ and joining c to $\ell_b, c_2, \ldots, c_{\deg(v)-2}$. Note that $\deg_{T'}(v) = \deg_{T'}(\ell_b) = 2, \deg_{T'}(c) = \deg(c) + \deg(v) - 3 \geq \deg(c) + 1 \geq 3$, while all other vertices have the same degree in T' as in T. On the one hand, if $\deg(c) = 2$, then $s(T') = s(T) - \deg(v) + \deg_{T'}(c) = s(T) - 1$. On the other hand, if $\deg(c) \geq 3$, then $s(T') = s(T) - \deg(v) + \deg(v) - 3 = s(T) - 3$. Then S is a **TRDS** of T'. As $T' \in \mathcal{T}$ and $s(T') < s(T)$, we obtain a contradiction in both cases. Thus, $\deg(v) = 3$. $\diamond$

Claim 2 *No two vertices of degree 3 are adjacent.*

Proof. Using the notation employed in Claim 1, b is the only neighbor of v in S. By Claim 1, $\deg(b) \leq 2$. If $\deg(c) = 3$, then, by Claim 1, c is adjacent to a vertex in $V - S$ (other than v). Let T' be the (v, c, b)-pruning of T. Then S is a **TRDS** of T', and so, by definition of γ_{tr}, we have that $\gamma_{tr} \leq \gamma_{tr}(T') \leq |S| = \gamma_{tr}$. Hence, $T' \in \mathcal{T}$. However, as $s(T') < s(T)$, we obtain a contradiction. $\diamond$

Using the notation employed in the proof of Claim 1, the vertex $b \in S$ and, as it must be adjacent to another vertex in S, $\deg(b) = 2$ (cf. Claim 1). Let $b' \in S$ be the vertex adjacent to b and suppose b' is not a leaf. Then, by Claim 1, $\deg(b') = 2$. Let b'' be the neighbor of b' different from b. Then S is a **TRDS** of a tree T' obtained from T by deleting the edge $b'b''$ and joining the vertex b'' to some leaf of the component of $T - v$ containing c. Thus $T' \in \mathcal{T}$ and b' is a leaf of T'. Hence we may assume that b' is a leaf of T.

By Claim 2, $\deg(a) = \deg(c) = 2$. Let a'(c', respectively) be the neighbor of a (c, respectively) which is different from v. Necessarily, $a', c' \in S$. Then $\deg(a') = \deg(c') = 2$ (cf. Claim 1). As each vertex in S is adjacent to another vertex of S, there exist vertices a'' and c'' in S which are adjacent to a' and c' respectively. We may assume, as we did for b', that a'' is a leaf of T.

If $n = 9$, then $\gamma_{tr}(T) = 6 = \lceil \frac{n+2}{2} \rceil$. Suppose, therefore, that $n \geq 10$. Let T' be the component of $T - cc'$ containing c'. Then $S \cap V(T')$ is a **TRDS** of T', so that $|S \cap V(T')| \geq \gamma_{tr}(T')$. Hence, $|S| \geq 4 + \gamma_{tr}(T')$. Applying the inductive hypothesis to the tree T' of order $n - 7$, we have $\gamma_{tr}(T') \geq \lceil \frac{n-5}{2} \rceil$, and so $\gamma_{tr}(T) = |S| \geq \lceil \frac{n+3}{2} \rceil \geq \lceil \frac{n+2}{2} \rceil$. □

3 Extremal trees T with $\gamma_{tr}(T) = \lceil \frac{n(T)+2}{2} \rceil$

Let $\mathcal{T}$ be the class of all trees T of order $n(T)$ such that $\gamma_{tr}(T) = \lceil \frac{n(T)+2}{2} \rceil$. We will constructively characterize the trees in $\mathcal{T}$. In order to state the characterization, we define four simple operations on a tree T.

O1. Join a leaf or a remote vertex of T to a vertex of K_1, where $n(T)$ is even.

O2. Join a vertex v of T which lies on an endpath vxz to a leaf of P_3, where $n(T)$ is even.

O3. Join a vertex v of T which lies on an endpath vx_1x_2z to a leaf of P_3, where $n(T)$ is even.

O4. Join a remote vertex or a leaf of T to a leaf of each of ℓ disjoint copies of P_4 for some $\ell \geq 1$.

Let $\mathcal{C}$ be the class of all trees obtained from P_2 by a finite sequence of Operations **O1- O4**.

We will show that $T \in \mathcal{T}$ if and only if $T \in \mathcal{C}$.

Lemma 4 *Let $T' \in \mathcal{T}$ be a tree of even order $n(T')$. If T is obtained from T' by one of the Operations* **O1-O3***, then $T \in \mathcal{T}$.*

Proof. Let S be a $\gamma_{tr}(T')$-set of T' throughout the proof of this result.

Case 1. T is obtained from T' by Operation **O1**.

Let u be a leaf or a remote vertex of T', and suppose T is formed by attaching the singleton v to u. Then $S \cup \{v\}$ is a **TRDS** set of T, and so $\lceil \frac{n(T')+3}{2} \rceil \leq \gamma_{tr}(T) \leq \lceil \frac{n(T')+2}{2} \rceil + 1$. Since $n(T')$ is even, we have $\gamma_{tr}(T) = \lceil \frac{n(T)+2}{2} \rceil$. Thus, $T \in \mathcal{T}$.

Case 2. T is obtained from T' by Operation **O2** or Operation **O3**.

Suppose v lies on the endpath vxz or vx_1x_2z and T is obtained from T' by adding the path y_1y_2z' to T' and joining y_1 to v.

We show that $v \notin S$. First consider the case when v lies on the endpath vxz. Suppose $v \in S$. Then $S' = S - \{z\}$ is a **TRDS** of $T'' = T' - \{z\}$, and so $\lceil \frac{n(T')+1}{2} \rceil \leq \gamma_{tr}(T'') \leq \lceil \frac{n(T')+2}{2} \rceil - 1$. However, as $n(T')$ is even, we have $\frac{n(T')+2}{2} \leq \gamma_{tr}(T'') \leq \frac{n(T')+2}{2} - 1$, which is a contradiction. Thus, $v \notin S$.

In the case when v lies on the endpath vx_1x_2z, one may show, as in the previous paragraph, that $x_1 \notin S$. But then $v \notin S$, as required.

In both cases, the set $S \cup \{y_2, z'\}$ is a **TRDS** of T, and so $\lceil \frac{n(T')+5}{2} \rceil \leq \gamma_{tr}(T) \leq \lceil \frac{n(T')+2}{2} \rceil + 2$. However, as $n(T')$ is even, we have $\gamma_{tr}(T) = \frac{n(T')+6}{2} = \lceil \frac{n(T)+2}{2} \rceil$. Thus, $T \in \mathcal{T}$.

The proof is complete. □

Lemma 5 *Let $T' \in \mathcal{T}$ be a tree of order $n(T')$. If T is obtained from T' by the Operation* **O4***, then $T \in \mathcal{T}$.*

Proof. Let S be a $\gamma_{tr}(T')$-set of T', and suppose v is a remote vertex or a leaf of T'. Then $v \in S$. Let T be the tree which is obtained from T' by adding the paths $u_ix_iy_iz_i$ to T' and joining u_i to v for $i = 1, \ldots, \ell$. Then $S \cup_{i=1}^{\ell} \{y_i, z_i\}$ is a **TRDS** of T, and so $\lceil \frac{n(T')+4\ell+2}{2} \rceil \leq \gamma_{tr}(T) \leq \lceil \frac{n(T')+2}{2} \rceil + 2\ell$. Consequently, $\gamma_{tr}(T) = \lceil \frac{n(T)+2}{2} \rceil$, and so $T \in \mathcal{T}$. □

We are now in a position to prove the main result of this section.

Theorem 6 *T is in $\mathcal{C}$ if and only if T is in $\mathcal{T}$.*

Proof. Assume $T \in \mathcal{C}$. We show that $T \in \mathcal{T}$, by using induction on $c(T)$, the number of operations required to construct the tree T. If $c(T) = 0$, then $T = P_2$, which is in $\mathcal{T}$. Assume, then, for all trees $T' \in \mathcal{C}$ with $c(T') < k$, where $k \geq 1$ is an integer, that T' is in $\mathcal{T}$. Let $T \in \mathcal{C}$ be a tree with $c(T) = k$. Then T is obtained from some tree T' by one of the Operations **O1** − **O4**. But then $T' \in \mathcal{C}$ and $c(T') < k$. Applying the inductive hypothesis to T', T' is in $\mathcal{T}$. Hence, by Lemma 4 or Lemma 5 , T is in $\mathcal{T}$.

To show that $T \in \mathcal{C}$ for a nontrivial $T \in \mathcal{T}$, we use induction on n, the order of the tree T. If $n = 2$, then $T = P_2 \in \mathcal{C}$. Let $T \in \mathcal{T}$ be a tree of order $n \geq 3$, and assume for all trees $T' \in \mathcal{T}$ of order $2 \leq n(T') < n$, that $T' \in \mathcal{C}$. Since $n(T) \geq 3$, $\text{diam}(T) \geq 2$.

If $\text{diam}(T) = 2$, then T is a star with exactly two leaves, which can be constructed from P_2 by applying Operation **O1**. Thus, $T \in \mathcal{C}$.

Since no double star is in $\mathcal{T}$, we may assume $\text{diam}(T) \geq 4$. Throughout S will be used to denote a $\gamma_{tr}(T)$-set of T.

Claim 3 *Let z be a leaf of T. If $S - \{z\}$ is a* **TRDS** *of $T' = T - z$, then $T \in \mathcal{C}$.*

Proof. Assume $S - \{z\}$ is a **TRDS** of T'. Then $\lceil \frac{n-1+2}{2} \rceil \leq \gamma_{tr}(T') \leq \lceil \frac{n+2}{2} \rceil - 1$. This yields a contradiction when n is even. Hence, n is odd, and $\gamma_{tr}(T') = \frac{n+1}{2} = \lceil \frac{n(T')+2}{2} \rceil$.

Thus, $T' \in \mathcal{T}$, with $n(T') = n - 1$ even. By the induction assumption, $T' \in \mathcal{C}$. The tree T can now be constructed from T' by applying Operation **O1**, whence $T \in \mathcal{C}$. $\diamond$

Claim 3 implies that if vxz is an endpath of T, then we may assume $v \notin S$, since otherwise the tree is constructible. Claim 3 also implies that every remote vertex of T is adjacent to exactly one leaf, since otherwise it is constructible.

Claim 4 *If u is a leaf of T and v is either another leaf of T or the remote vertex adjacent to u, then $S' = S - \{u, v\}$ is not a* **TRDS** *of $T' = T - u - v$.*

Proof. Suppose, to the contrary, that S' is a **TRDS** of T'. Then $\lceil \frac{n-2+2}{2} \rceil \le \gamma_{tr}(T') \le \lceil \frac{n+2}{2} \rceil - 2$. Thus, $\lceil \frac{n}{2} \rceil + 2 \le \lceil \frac{n+2}{2} \rceil$, which yields a contradiction. $\diamond$

Let T be rooted at a leaf r of a longest path.

Let v be any vertex on a longest path at distance $\text{diam}(T) - 2$ from r. Suppose v lies on the endpath vyz'. Then, by the remark above, $v \notin S$, which implies that v is not adjacent to a leaf. If v also lies on the endpath vxz, then $S - \{x, z\}$ is a **TRDS** of $T - x - z$, which is a contradiction by Claim 4.

Thus, we assume each vertex on a longest path at distance $\text{diam}(T) - 2$ or $\text{diam}(T) - 1$ from r has degree two.

Let v be any vertex on a longest path at distance $\text{diam}(T) - 3$ from r. Let vy_1y_2z' be an endpath of T. Then $y_1 \notin S$, and so $v \notin S$, which means all neighbors of v have degree at least 2.

Assume v also lies on the path vxz, where z is a leaf. Then, since each remote vertex is adjacent to exactly one leaf, vxz is an endpath. If v is dominated by a vertex other than x, then $S - \{x, z\}$ is a **TRDS** of $T' = T - x - z$, which is a contradiction (cf. Claim 4). Hence, v is dominated only by x. Then $S' = S - \{y_2, z'\}$ is a **TRDS** of $T' = T - y_1 - y_2 - z'$ and so $\lceil \frac{n-3+2}{2} \rceil \le \gamma_{tr}(T') \le \lceil \frac{n+2}{2} \rceil - 2$. This yields a contradiction when n is even. Hence, n is odd and $\gamma_{tr}(T') = \frac{n-1}{2} = \lceil \frac{n(T')+2}{2} \rceil$. Thus, $T' \in \mathcal{T}$, with $n(T') = n - 3$ even. By the induction assumption, $T' \in \mathcal{C}$. The tree T can now be constructed from T' by applying Operation **O2**, whence $T \in \mathcal{C}$.

Assume v lies on the path vx_1x_2z. Since x_1 (x_2, respectively) is on a longest path at distance $\text{diam}(T) - 2$ ($\text{diam}(T) - 1$, respectively) from r, we have $\deg(x_1) = 2$ ($\deg(x_2) = 2$, respectively). This implies that vx_1x_2z is an endpath, and so $x_1 \notin S$. But then $S' = S - \{x_2, z\}$ is a **TRDS** of $T' = T - x_1 - x_2 - z$. Thus, $\lceil \frac{n-3+2}{2} \rceil \le \gamma_{tr}(T') \le \lceil \frac{n+2}{2} \rceil - 2$. This yields a contradiction when n is even. Hence, n is odd and $\gamma_{tr}(T') = \lceil \frac{n(T')+2}{2} \rceil$. Thus, $T' \in \mathcal{T}$, with $n(T') = n - 3$ even. By the induction assumption, $T' \in \mathcal{C}$ and T can now be constructed from T' by applying Operation **O3**, whence $T \in \mathcal{C}$.

Thus, we assume each vertex on a longest path at distance $\text{diam}(T) - 3$ from r has degree two.

Let v be any vertex on a longest path at distance $\text{diam}(T) - 4$ from r. As $P_5 \notin \mathcal{T}$, $v \neq r$

and $\operatorname{diam}(T) \geq 5$.

Assume $\deg_T(v) \geq 3$. Let $vy_1y_2y_3z'$ be an endpath of T. But then, as y_2y_3z' is an endpath of T, it follows that $y_2 \notin S$, which implies $y_1 \notin S$ and $v \in S$. Moreover, $S' = S - \{y_3, z'\}$ is a **TRDS** of $T' = T - y_1 - y_2 - y_3 - z'$. Thus, $\lceil \frac{n-4+2}{2} \rceil \leq \gamma_{tr}(T') \leq \lceil \frac{n+2}{2} \rceil - 2$, whence $\gamma_{tr}(T') = \lceil \frac{n(T')+2}{2} \rceil$. We conclude that $T' \in \mathcal{T}$, and by the induction assumption, $T' \in \mathcal{C}$. If $\deg_T(v) = 2$ or when v is a remote vertex, then T can be constructed from T' by applying Operation **O4**.

We therefore assume that $\deg_T(v) \geq 3$ and that v is not adjacent to a leaf.

If v also lies on the path vxz, where z is a leaf, then $v \notin S$, which is a contradiction.

We now suppose v lies on the path vx_1x_2z, where z is a leaf. Then, since x_2 is a remote vertex, we have $\deg(x_2) = 2$. As x_1x_2z is an endpath of T, it follows that $x_1 \notin S$. As x_1 must be adjacent to another vertex in $V - S$, vertex x_1 lies on a path x_1, u_1, u_2, z''. But then x_1, with $\deg(x_1) \geq 3$, is a vertex at distance $\operatorname{diam}(T) - 3$ on a longest path from r, which is a contradiction.

Let e be the edge that joins v with its parent, and let $T(v)$ be the component of $T - e$ that contains v. Then $T(v)$ consists of ℓ disjoint paths $u_ix_iy_iz_i$ $(i = 1, \ldots, \ell)$ with v joined to u_i for $i = 1, \ldots, \ell$. Let $i \in \{1, \ldots, \ell\}$. Since $x_iy_iz_i$ is an endpath of T, we have $x_i \notin S$, $u_i \notin S$ and $v \in S$. Then $S - \cup_{i=1}^{\ell}\{y_i, z_i\}$ is a **TRDS** of $T' = T - (T(v) - v)$, and so $\lceil \frac{n-4\ell+2}{2} \rceil \leq \gamma_{tr}(T') \leq \lceil \frac{n+2}{2} \rceil - 2\ell$, whence $\gamma_{tr}(T') = \lceil \frac{n(T')+2}{2} \rceil$. Thus, $T' \in \mathcal{T}$, and by the induction assumption, $T' \in \mathcal{C}$. Note that v is a leaf of T'. The tree T can now be constructed from T' by applying Operation **O4**, whence $T \in \mathcal{C}$. $\Box$

Theorem 7 *Let T be a tree of order $n(T)$. If $n(T) \equiv 0 \bmod 4$, then $\gamma_{tr}(T) \geq \lceil \frac{n(T)+2}{2} \rceil + 1$.*

Proof. We will show that every tree T in $\mathcal{T} = \mathcal{C}$ has $n(T) \not\equiv 0 \bmod 4$, by using induction on $s(T)$, the number of operations required to construct the tree T. If $s(T) = 0$, then $T = P_2$, and $2 \not\equiv 0 \bmod 4$. Assume, then, for all trees $T' \in \mathcal{C}$ with $s(T') < k$, where $k \geq 1$ is an integer, that $n(T') \not\equiv 0 \bmod 4$. Let $T \in \mathcal{C}$ be a tree with $s(T) = k$. Then T is obtained from some tree T' by one of the Operations **O1** $-$ **O4**. Then $T' \in \mathcal{C}$, and by the induction hypothesis, $n(T') \not\equiv 0 \bmod 4$. If T is obtained from T' by one of the Operations **O1** $-$ **O3**, then $n(T') \equiv 2 \bmod 4$, and, since either a path of order one or a path of order three is attached to T' to form T, $n(T) \not\equiv 0 \bmod 4$. Moreover, $n(T) = n(T') + 4$ if T is obtained from T' by Operation **O4**, whence $n(T) \not\equiv 0 \bmod 4$. The result now follows. $\Box$

4 Extremal trees T of order $n(T) \equiv 0 \bmod 4$ with $\gamma_{tr}(T) = \lceil \frac{n(T)+2}{2} \rceil + 1$

Let $\mathcal{T}^* = \{T \,|\, T$ is a tree of order $n(T) \equiv 0 \bmod 4$ such that $\gamma_{tr}(T) = \lceil \frac{n+2}{2} \rceil + 1\}$. In order to constructively characterize the trees in $\mathcal{T}^*$, we define the following operations on a tree T:

O5. Join a leaf or a remote vertex v of T to a vertex of K_1, where $n(T) \equiv 3 \bmod 4$.

O6. Join a vertex v of T which lies on an endpath vxz to a vertex of K_2, where $n(T) \equiv 2 \bmod 4$.

O7. Join a vertex v of T which lies on an endpath vx_1x_2z to a vertex of K_2, where $n(T) \equiv 2 \bmod 4$.

O8. Join a vertex v of T which lies on an endpath vxz to a leaf of P_3, where $n(T) \equiv 1 \bmod 4$.

O9. Join a vertex v of T which lies on an endpath vx_1x_2z to a leaf of P_3, where $n(T) \equiv 1 \bmod 4$.

Let $\mathcal{I} = \{T \,|\, T$ is a tree obtained by applying one of the Operations **O5** − **O9** to a tree $T' \in \mathcal{C}$ exactly once$\}$. Let $\mathcal{C}^* = \{T \,|\, T$ is a tree obtained from a tree $T' \in \mathcal{I}$ by applying Operation **O4** to T' zero or more times$\}$. We will show that $\mathcal{T}^* = \mathcal{C}^*$.

Lemma 8 *Let $T' \in \mathcal{C}$ be a tree of order $n(T') \equiv 3 \bmod 4$. If T is obtained from T' by Operation* **O5***, then $T \in \mathcal{T}^*$.*

Proof. Let u be a leaf or a remote vertex of T', and suppose T is formed by attaching the singleton v to u. Let S be a $\gamma_{tr}(T')$-set of T'. Then $S \cup \{v\}$ is a **TRDS** set of T, and so, since $n(T) \equiv 0 \bmod 4$, $\lceil \frac{n(T)+2}{2} \rceil + 1 \leq \gamma_{tr}(T) \leq |S| + 1 = \lceil \frac{n(T')+2}{2} \rceil + 1 = \lceil \frac{n(T)+1}{2} \rceil + 1$. Hence, $\gamma_{tr}(T) = \lceil \frac{n(T)+2}{2} \rceil + 1$, and so $T \in \mathcal{T}^*$. $\Box$

Lemma 9 *Let $T' \in \mathcal{C}$ be a tree of order $n(T') \equiv 2 \bmod 4$. If T is obtained from T' by either Operation* **O6** *or Operation* **O7***, then $T \in \mathcal{T}^*$.*

Proof. Let $\{u, v\}$ be the vertex set of K_2 and let S be a $\gamma_{tr}(T')$-set. The set $S \cup \{u, v\}$ is a **TRDS** of T, and so, since $n(T) \equiv 0 \bmod 4$, $\lceil \frac{n(T)+2}{2} \rceil + 1 \leq \gamma_{tr}(T) \leq |S| + 2 = \lceil \frac{n(T')+2}{2} \rceil + 2 = \lceil \frac{n(T)}{2} \rceil + 2$. Hence, $\gamma_{tr}(T) = \lceil \frac{n(T)+2}{2} \rceil + 1$, and so $T \in \mathcal{T}^*$. $\Box$

Lemma 10 *Let $T' \in \mathcal{C}$ be a tree of order $n(T') \equiv 1 \bmod 4$. If T is obtained from T' by either Operation* **O8** *or Operation* **O9***, then $T \in \mathcal{T}^*$.*

Proof. Let S be a $\gamma_{tr}(T')$-set of T'. Assume v lies on the endpath vxz or vx_1x_2z and T is obtained from T' by adding the path y_1y_2z' to T' and joining y_1 to v. We show that $v \notin S$.

First consider the case when v lies on the endpath vxz. Suppose $v \in S$. Then $x, z \in S$, and $S - \{z\}$ is **TRDS** of $T'' = T' - z$. Since $n(T'') \equiv 0 \bmod 4$, $\lceil \frac{n(T'')+2}{2} \rceil + 1 \leq \gamma_{tr}(T'') \leq |S| - 1 = \lceil \frac{n(T')+2}{2} \rceil - 1 = \lceil \frac{n(T'')+3}{2} \rceil - 1$, and so $\frac{n(T'')+4}{2} \leq \frac{n(T'')+2}{2}$, which is a contradiction. Thus, $v \notin S$.

In the case when v lies on the endpath vx_1x_2z, one may show, as in the previous paragraph, that $x_1 \notin S$. But then $v \notin S$, as required.

In both cases, the set $S \cup \{y_2, z'\}$ forms a **TRDS** of T, so that $\lceil \frac{n(T)+2}{2} \rceil + 1 \le \gamma_{tr}(T) \le |S| + 2 = \lceil \frac{n(T')+2}{2} \rceil + 2 = \lceil \frac{n(T)-1}{2} \rceil + 2$. Hence, $\gamma_{tr}(T) = \lceil \frac{n(T)+2}{2} \rceil + 1$, and so $T \in \mathcal{T}^*$. $\square$

The proof of the following result is similar to that of Lemma 5.

Lemma 11 *If T is obtained from $T' \in \mathcal{T}^*$ by Operation* **O4***, then $T \in \mathcal{T}^*$.*

Lemma 12 *If T is in $\mathcal{I}$, then T is in $\mathcal{T}^*$.*

Proof. Assume $T \in \mathcal{I}$. Then T is obtained from $T' \in \mathcal{C}$ by applying one of the Operations **O5** $-$ **O9** exactly once. Then, by Lemmas 8, 9 and 10, $T \in \mathcal{T}^*$. $\square$

Theorem 13 *T is in $\mathcal{C}^*$ if and only if T is in $\mathcal{T}^*$.*

Proof. Assume $T \in \mathcal{C}^*$. We show that $T \in \mathcal{T}^*$, by using induction on $c(T)$, the number of operations required to construct the tree T. If $c(T) = 0$, then $T \in \mathcal{I}$, and the result follows from Lemma 12. Assume, then, for all trees $T' \in \mathcal{C}^*$ with $c(T') < k$, where $k \ge 1$ is an integer, that T' is in $\mathcal{T}^*$. Let $T \in \mathcal{C}^*$ be a tree with $c(T) = k$. Then T is obtained from some tree T' by applying Operation **O4**. But then $T' \in \mathcal{C}^*$ and $c(T') < k$. Applying the inductive hypothesis to T', T' is in $\mathcal{T}^*$. Hence, by Lemma 11, T is in $\mathcal{T}^*$.

To show that $T \in \mathcal{C}^*$ for a nontrivial $T \in \mathcal{T}^*$, we employ induction on $4n$, the order of the tree T. Suppose $n = 1$. Then $T \cong K_{1,3}$ or $T \cong P_4$, and T can be constructed from $P_3 \in \mathcal{C}$ by applying Operation **O5**.

Let $T \in \mathcal{T}^*$ be a tree of order $4n$, where $n \ge 2$, and suppose $T' \in \mathcal{C}^*$ for all trees $T' \in \mathcal{T}^*$ of order $4n'$ where $n' < n$.

The only trees T with $\mathrm{diam}(T) \le 3$ which are in $\mathcal{T}^*$ are $K_{1,3}$ and P_4. As $4n \ge 8$, it follows that $\mathrm{diam}(T) \ge 4$. Throughout S will be used to denote a γ_{tr}-set of T, i.e. $|S| = \lceil \frac{n+2}{2} \rceil + 1$.

Claim 5 *If u and v are vertices of T such that $T' = T - u - v$ is a tree and $S' = S - \{u, v\}$ is a* **TRDS** *of T', then $n(T') \equiv 2 \bmod 4$ and $T' \in \mathcal{C}$.*

Proof. As $\lceil \frac{n-2+2}{2} \rceil \le \gamma_{tr}(T') \le \lceil \frac{n+2}{2} \rceil + 1 - 2$, we have $\gamma_{tr}(T') = \lceil \frac{n-2+2}{2} \rceil = \lceil \frac{n(T')+2}{2} \rceil$, and so $T' \in \mathcal{C}$. $\diamond$

Claim 6 *Let z be a leaf of T. If $S - \{z\}$ is a* **TRDS** *of $T' = T - z$, then $T \in \mathcal{C}^*$.*

Proof. Assume $S - \{z\}$ is a **TRDS** of T'. Then $\lceil \frac{n-1+2}{2} \rceil \le \gamma_{tr}(T') \le \lceil \frac{n+2}{2} \rceil + 1 - 1 = \lceil \frac{n+2}{2} \rceil$. Hence, $n - 1 \equiv 3 \bmod 4$ and $\gamma_{tr}(T') = \lceil \frac{n+1}{2} \rceil = \lceil \frac{n(T')+2}{2} \rceil$. Thus, $T' \in \mathcal{C}$. The tree T can now be constructed from T' by applying Operation **O5**, whence $T \in \mathcal{C}^*$. $\diamond$

Claim 6 implies that if vxz is an endpath of T, then we may assume $v \notin S$, since otherwise the tree is constructible. Claim 6 also implies that every remote vertex of T is adjacent to exactly one leaf, since otherwise it is constructible.

Let T be rooted at a leaf r of a longest path.

Let v be any vertex on a longest path at distance $\text{diam}(T) - 2$ from r. Suppose v lies on the endpath vyz'. Then, by the remark above, $v \notin S$, which implies that v is not adjacent to a leaf. If v also lies on the endpath vxz, then $S - \{x, z\}$ is a **TRDS** of $T - x - z$ and so $T' \in \mathcal{C}$ (cf. Claim 5), whence $T \in \mathcal{C}^*$ (as it can be constructed from T' by applying Operation **O6**).

Thus, we assume each vertex on a longest path at distance $\text{diam}(T) - 2$ or $\text{diam}(T) - 1$ from r has degree two.

Let v be any vertex on a longest path at distance $\text{diam}(T) - 3$ from r. Let vy_1y_2z' be an endpath of T. Then $y_1 \notin S$, and so $v \notin S$, which means all neighbors of v have degree at least 2.

Assume v also lies on the path vxz, where z is a leaf. Then, since each remote vertex is adjacent to exactly one leaf, vxz is an endpath. If v is dominated by a vertex other than x, then $S - \{x, z\}$ is a **TRDS** of $T' = T - x - z$ and so $T' \in \mathcal{C}$ (cf. Claim 5), whence $T \in \mathcal{C}^*$ (as it can be constructed from T' by applying Operation **O7**). Hence, v is dominated only by x. Then $S' = S - \{y_2, z'\}$ is a **TRDS** of $T' = T - y_1 - y_2 - z'$ and so $\lceil \frac{n-3+2}{2} \rceil \leq \gamma_{tr}(T') \leq \lceil \frac{n+2}{2} \rceil - 1$. But then $\gamma_{tr}(T') = \lceil \frac{n-1}{2} \rceil = \lceil \frac{n(T')+2}{2} \rceil$. Thus, $T' \in \mathcal{C}$. The tree T can now be constructed from T' by applying Operation **O8**.

Assume v lies on the path vx_1x_2z. Since x_1 (x_2, respectively) is on a longest path at distance $\text{diam}(T) - 2$ ($\text{diam}(T) - 1$, respectively) from r, we have $\deg(x_1) = 2$ ($\deg(x_2) = 2$, respectively). This implies that vx_1x_2z is an endpath, and so $x_1 \notin S$. But then $S' = S - \{x_2, z\}$ is a **TRDS** of $T' = T - x_1 - x_2 - z$. Thus, $\lceil \frac{n-3+2}{2} \rceil \leq \gamma_{tr}(T') \leq \lceil \frac{n+2}{2} \rceil - 1$. But then $\gamma_{tr}(T') = \lceil \frac{n-1}{2} \rceil = \lceil \frac{n(T')+2}{2} \rceil$. Thus, $T' \in \mathcal{C}$ and so T can now be constructed from T' by applying Operation **O9**.

Thus, we assume each vertex on a longest path at distance $\text{diam}(T) - 3$ from r has degree two.

Let v be any vertex on a longest path at distance $\text{diam}(T) - 4$ from r. As $P_5 \notin \mathcal{T}^*$, $v \neq r$ and $\text{diam}(T) \geq 5$.

Assume $\deg_T(v) \geq 3$. Let $vy_1y_2y_3z'$ be an endpath of T. But then, as y_2y_3z' is an endpath of T, it follows that $y_2 \notin S$, which implies $y_1 \notin S$ and $v \in S$. Moreover, $S' = S - \{y_3, z'\}$ is a **TRDS** of $T' = T - y_1 - y_2 - y_3 - z'$. Thus, $\lceil \frac{n-4+2}{2} \rceil + 1 \leq \gamma_{tr}(T') \leq \lceil \frac{n+2}{2} \rceil - 1$, whence $\gamma_{tr}(T') = \lceil \frac{n(T')+2}{2} \rceil + 1$. We conclude that $T' \in \mathcal{T}^*$, and by the induction assumption, $T' \in \mathcal{C}^*$. If $\deg_T(v) = 2$ or when v is a remote vertex, then T can be constructed from T' by applying Operation **O4**, whence $T \in \mathcal{C}^*$.

We therefore assume that $\deg_T(v) \geq 3$ and that v is not adjacent to a leaf.

If v also lies on the path vxz, where z is a leaf, then $v \notin S$, which is a contradiction.

We now suppose v lies on the path vx_1x_2z, where z is a leaf. Then, since x_2 is a remote vertex, we have $\deg(x_2) = 2$. As x_1x_2z is an endpath of T, it follows that $x_1 \notin S$. As x_1 must be adjacent to another vertex in $V - S$, vertex x_1 lies on a path x_1, u_1, u_2, z''. But then x_1, with $\deg(x_1) \geq 3$, is a vertex at distance $\mathrm{diam}(T) - 3$ on a longest path from r, which is a contradiction.

Let e be the edge that joins v with its parent, and let $T(v)$ be the component of $T - e$ that contains v. Then $T(v)$ consists of ℓ disjoint paths $u_ix_iy_iz_i$ $(i = 1, \ldots, \ell)$ with v joined to u_i for $i = 1, \ldots, \ell$. Let $i \in \{1, \ldots, \ell\}$. Since $x_iy_iz_i$ is an endpath of T, we have $x_i \notin S$, $u_i \notin S$ and $v \in S$. Then $S - \cup_{i=1}^{\ell}\{y_i, z_i\}$ is a **TRDS** of $T' = T - (T(v) - v)$, and so $\lceil \frac{n-4\ell+2}{2} \rceil + 1 \leq \gamma_{tr}(T') \leq \lceil \frac{n+2}{2} \rceil - 2\ell + 1$, whence $\gamma_{tr}(T') = \lceil \frac{n(T')+2}{2} \rceil + 1$. Thus, $T' \in \mathcal{T}^*$, and by the induction assumption, $T' \in \mathcal{C}^*$. Note that v is a leaf of T'. The tree T can now be constructed from T' by applying Operation **O4**, whence $T \in \mathcal{C}^*$. $\square$

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