A Note on Restrained Domination in Trees

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Abstract

Let $G = (V, E)$ be a graph. A set $S \subseteq V$ is a restrained dominating set if every vertex not in S is adjacent to a vertex in S and to a vertex in $V - S$. The restrained domination number of G, denoted by $\gamma_r(G)$, is the smallest cardinality of a restrained dominating set of G. It is known that if T is a tree of order n, then $\gamma_r(T) \geq \lceil (n+2)/3 \rceil$. In this note we provide a simple constructive characterization of the extremal trees T of order n achieving this lower bound.

1 Introduction

In this paper, we follow the notation of [1]. Specifically, let $G = (V, E)$ be a graph with vertex set V and edge set E. Moreover, the notation P_n will denote the path of order n. A set $S \subseteq V$ is a *dominating set* of G if every vertex not in S is adjacent to a vertex in S. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set. The concept of domination in graphs, with its many variations, is now well studied in graph theory. The recent book of Chartrand and Lesniak [1] includes a chapter on domination. A thorough study of domination appears in [5, 6].

In this paper, we continue the study of a variation of the domination theme, namely that of restrained domination [2, 3, 4, 7, 8]. A set $S \subseteq V$ is a *restrained dominating set* if every vertex not in S is adjacent to a vertex in S and to a vertex in $V - S$. Every graph has a restrained dominating set, since $S = V$ is such a set. The *restrained domination number* of G, denoted by $\gamma_r(G)$, is the minimum cardinality of a restrained dominating set of G. A restrained dominating set of cardinality $\gamma_r(G)$ will be called a $\gamma_r(G) - set$.

The concept of restrained domination was introduced by Telle and Proskurowski [8], albeit indirectly, as a vertex partitioning problem. Here conditions are imposed on a set S, the complementary set $V - S$ and on edges between the sets S and $V - S$. For example, if we require that every vertex in $V - S$ should be adjacent to some other vertex of $V - S$ (the condition on the set $V - S$) and to some vertex in S (the condition on edges between the sets S and $V - S$), then S is a restrained dominating set.

One application of domination is that of prisoners and guards. For security, each prisoner must be seen by some guard; the concept is that of domination. However, in order to protect the rights of prisoners, we may also require that each prisoner is seen by another prisoner; the concept is that of restrained domination.

It is known [2] that if T is a tree of order n, then $\gamma_r(T) \geq \lceil (n+2)/3 \rceil$.

We refer to a vertex of degree 1 in T as a *leaf* of T. A vertex adjacent to a leaf we call a *remote vertex* of T. For a vertex v of T, we shall use the expression, *attach a* P_m *at* v, to refer to the operation of taking the union of T and a path P_m and joining one of the ends of this path to v with an edge.

For $n \geq 1$, let $\mathcal{T}_n = \{T \mid T$ is a tree of order n such that $\gamma_r(T) = \lceil (n+2)/3 \rceil \}$. A constructive characterization of the extremal trees T of order n achieving this lower bound were characterized in [2]. For the purpose of stating this characterization, we define a **type (1)** operation on a tree T as attaching a P_2 at v where v is a vertex of T not belonging to some minimum restrained dominating set of T, and a **type (2)** operation as attaching a P_3 at v where v belongs to some minimum restrained dominating set of T. For $i = 1, 2$, let T_i be the tree obtained from $K(1,3)$ by subdividing i edges once.

Let $\mathcal{C}_{3k} = \{T \mid T$ is a tree of order $3k$ which can be obtained from the tree T_2 by a finite sequence of operations of type (2)$\}$. Let $\mathcal{C}_{3k+1} = \{T \mid T$ is a tree of order $3k+1$ which can be obtained from P_4 by a finite sequence of operations of type (2)$\}$. Finally, let $\mathcal{C}_{3k+2} = \{T \mid T$ is a tree of order $3k+2$ which can be obtained from P_5 or from the tree T_1 by a finite sequence of operations of type (2)$\} \cup \{T \mid T$ is a tree of order $3k+2$ which can be constructed from the tree T_2 by a finite sequence of operations of type (2), followed by one operation of type (1) and then by a finite sequence of operations of type (2)$\}$.

It was established in [2] that

Theorem 1 *For* $n \geq 4$, $\mathcal{T}_n = \mathcal{C}_n$.

The purpose of this note is to provide a simpler constructive characterization of the extremal trees T of order n achieving this lower bound.

We denote the set of leaves of a tree T by $L(T)$. For $v \in V(T)$ and $\ell \in L(T)$, the path $vx_1 \ldots x_k\ell$ is called a $v-L$ *endpath* if $\deg x_i = 2$ for each i. If the vertex v need not be specified, a $v-L$ path is also called an *endpath*.

2 Extremal trees T with $\gamma_r(T) = \lceil (n+2)/3 \rceil$

Let $\mathcal{T}$ be the class of all trees T of order n such that $\gamma_r(T) = \lceil \frac{n+2}{3} \rceil$. We will constructively characterize the trees in $\mathcal{T}$. In order to state the characterization, we define three simple operations on a tree T.

O1. Join a leaf or a remote vertex, or a vertex v or x of T on an endpath $vxyz$ to a vertex of K_1, where $n(T) \equiv 1 \bmod 3$.

O2. Join a remote vertex, or a vertex v of T which lies on an endpath vxz to a leaf of P_2, where $n(T) \equiv 0 \bmod 3$ or $n(T) \equiv 1 \bmod 3$.

O3. Join a leaf of T to ℓ disjoint copies of P_3 for some $\ell \geq 1$.

Let $\mathcal{C}$ be the class of all trees obtained from P_2 or P_4 by a finite sequence of Operations **O1- O3**.

We will show that $T \in \mathcal{T}$ if and only if $T \in \mathcal{C}$.

Let S be a $\gamma_r(T')$-set of T' throughout the proofs of the following lemmas.

Lemma 2 *Let $T' \in \mathcal{T}$ be a tree of order $n \equiv 1 \bmod 3$. If T is obtained from T' by Operation* **O1***, then $T \in \mathcal{T}$.*

Proof. Let u be a leaf or a remote vertex, or a vertex w or x on an endpath $wxyz$ of T', and suppose T is formed by attaching the singleton v to u. Then $S \cup \{v\}$ is a **RDS** of T, and so $\lceil \frac{n+3}{3} \rceil \leq \gamma_r(T) \leq \lceil \frac{n+2}{3} \rceil + 1$. Since $n \equiv 1 \bmod 3$, we have $\gamma_r(T) = \lceil \frac{n(T)+2}{3} \rceil$. Thus, $T \in \mathcal{T}$. □

Lemma 3 *Let $T' \in \mathcal{T}$ be a tree of order $n \equiv 0 \bmod 3$ or $n \equiv 1 \bmod 3$. If T is obtained from T' by Operation* **O2***, then $T \in \mathcal{T}$.*

Proof. Suppose v is a remote vertex or v lies on the endpath vxz and T is obtained from T' by adding the path vyz'.

We show that $v \notin S$. First consider the case when v is a remote vertex adjacent to a leaf z. Suppose $v \in S$. Then $S' = S - \{z\}$ is a **RDS** of $T'' = T' - z$, and so $\lceil \frac{n+1}{3} \rceil \leq \gamma_r(T'') \leq \lceil \frac{n+2}{3} \rceil - 1$, which is a contradiction when $n \equiv 0 \bmod 3$ or $n \equiv 1 \bmod 3$. Thus, $v \notin S$.

In the case when v lies on the endpath vxz, one may show, as in the previous paragraph, that $x \notin S$. But then $v \notin S$, as required.

In both cases, the set $S \cup \{z'\}$ is a **RDS** of T, and so $\lceil \frac{n+4}{3} \rceil \leq \gamma_r(T) \leq \lceil \frac{n+2}{3} \rceil + 1$. However, as $n \equiv 0 \bmod 3$ or $n \equiv 1 \bmod 3$, we have $\gamma_r(T) = \lceil \frac{n+4}{3} \rceil = \lceil \frac{n(T)+2}{3} \rceil$. Thus, $T \in \mathcal{T}$.

The proof is complete. □

Lemma 4 *Let $T' \in \mathcal{T}$ be a tree of order n. If T is obtained from T' by the Operation* **O3***, then $T \in \mathcal{T}$.*

Proof. Let S be a $\gamma_r(T')$-set of T', and suppose v is a leaf of T'. Then $v \in S$. Let T be the tree which is obtained from T by adding the paths $vx_iy_iz_i$ for $i = 1, \ldots, \ell$. Then $S \cup_{i=1}^{\ell} \{z_i\}$

is a **RDS** of T, and so $\lceil\frac{n+3\ell+2}{3}\rceil \leq \gamma_r(T) \leq \lceil\frac{n+2}{3}\rceil + \ell$. Consequently, $\gamma_r(T) = \lceil\frac{n(T)+2}{3}\rceil$, and so $T \in \mathcal{T}$. $\Box$

We are now in a position to prove the main result of this section.

Theorem 5 *$T \in \mathcal{C}$ if and only if $T \in \mathcal{T}$.*

Proof. Suppose $T \in \mathcal{C}$. We show that $T \in \mathcal{T}$, by using induction on $s(T)$, the number of operations required to construct the tree T. If $s(T) = 0$, then $T = P_2$ or $T = P_4$, both of which are in $\mathcal{T}$. Assume, then, for all trees $T' \in \mathcal{C}$ with $s(T') < k$, where $k \geq 1$ is an integer, that T' is in $\mathcal{T}$. Let $T \in \mathcal{C}$ be a tree with $s(T) = k$. Then T is obtained from some tree T' by one of the Operations $\mathbf{O1} - \mathbf{O3}$. But then $T' \in \mathcal{C}$ and $s(T') < k$. Applying the inductive hypothesis to T', T' is in $\mathcal{T}$. Hence, by Lemmas 2,3 or 4, $T \in \mathcal{T}$.

To show that $T \in \mathcal{C}$ for a nontrivial $T \in \mathcal{T}$, we use induction on n, the order of the tree T. If $n = 2$, then $T = P_2 \in \mathcal{C}$. If $n = 3$, then $T \notin \mathcal{T}$. If $n = 4$, then either $T = P_4$ or T is a star. If T is a star then $T \notin \mathcal{T}$. If $T = P_4$ then $T \in \mathcal{C}$. Let $T \in \mathcal{T}$ be a tree of order $n \geq 5$, and assume for all trees $T' \in \mathcal{T}$ of order $4 \leq n' < n$, that $T' \in \mathcal{C}$. Since $n(T) \geq 5$ and no stars are in $\mathcal{T}$, $\mathrm{diam}(T) \geq 3$.

If $\mathrm{diam}(T) = 3$, then T is a double star of order 5, has a remote vertex adjacent to two leaves, and is therefore constructible from P_4 by **O1**, whence $T \in \mathcal{C}$. Thus, we may assume $\mathrm{diam}(T) \geq 4$.

Throughout S will be used to denote a $\gamma_r(T)$-set of T.

Claim 1 *Suppose z is a leaf of T. If $S - \{z\}$ is a* **RDS** *of $T' = T - z$, then $n(T') \equiv 1$ mod 3 and $T' \in \mathcal{C}$.*

Proof. Suppose $S - \{z\}$ is a **RDS** of T'. Then $\lceil\frac{n-1+2}{3}\rceil \leq \gamma_r(T') \leq \lceil\frac{n+2}{3}\rceil - 1$. This yields a contradiction when $n \equiv 0 \bmod 3$ or $n \equiv 1 \bmod 3$. Hence, $n \equiv 2 \bmod 3$, and $\gamma_r(T') = \frac{n+1}{3} = \lceil\frac{n(T')+2}{3}\rceil$. Thus, $T' \in \mathcal{T}$, with $n(T') = n - 1 \equiv 1$ mod 3. By the induction assumption, $T' \in \mathcal{C}$. $\diamond$

Suppose vxz or vz is an endpath of T. If $v, x \in S$, then $S - \{z\}$ is a **RDS** of $T' = T - z$. By Claim 1, the tree $T' = (T - z) \in \mathcal{C}$ and T can be constructed from T' by Operation **O1**. Thus, if vxz or vz is an endpath of T, we may assume $v, x \notin S$.

Suppose v is a remote vertex adjacent to at least two leaves, and let z be a leaf adjacent to v. Then $S - \{z\}$ is a **RDS** of $T' = T - z$. By Claim 1, the tree $T' = (T - z) \in \mathcal{C}$ and T can be constructed from T' by Operation **O1**. Thus, we may assume that every remote vertex is adjacent to exactly one leaf.

Let T be rooted at a leaf r of a longest path.

Let v be any vertex on a longest path at distance $\mathrm{diam}(T) - 2$ from r. Suppose v lies on the endpath vyz'. Then, by the above remark, $v, y \notin S$. Suppose $\deg(v) \geq 3$ and first assume v

is a remote vertex adjacent to a leaf u. Since diam(T) ≥ 4, v has a parent vertex v_0. Suppose $v_0 \in S$. If $\deg(v) \geq 4$, since, by Claim 1, v is adjacent to one leaf only, x is on an endpath vxz where $x \notin S$. Since $v_0 \in S$, it follows that $S' = S - \{u, z\}$ is a **RDS** for $T' = T - u - x - z$. Hence, $\lceil \frac{(n-3)+2}{3} \rceil \leq \gamma_r(T') \leq \lceil \frac{n+2}{3} \rceil - 2$, which is a contradiction. Hence $\deg(v) = 3$. Consider $T' = T - u$. The vertex v in T' is on the endpath v_0vyz'. Since $v_0 \in S$, it follows that $S' = S - \{u\}$ is a **RDS** for T'. Thus, by Claim 1, $T' \in \mathcal{C}$ and T can be constructed from T' by Operation **O1**, whence $T \in \mathcal{C}$. So suppose $v_0 \notin S$. Then $S' = S - \{z'\}$ is a **RDS** for $T' = T - y - z'$. Hence, $\lceil \frac{(n-2)+2}{3} \rceil \leq \gamma_r(T') \leq \lceil \frac{n+2}{3} \rceil - 1$, which is a contradiction when $n \equiv 1 \bmod 3$. Hence $n \equiv 0 \bmod 3$ or $n \equiv 2 \bmod 3$ and $\gamma_r(T') = \lceil \frac{n}{3} \rceil = \lceil \frac{n(T')+2}{3} \rceil$. Thus, $T' \in \mathcal{T}$, with $n(T') = n - 2 \equiv 0 \bmod 3$ or $n(T') = n - 2 \equiv 1 \bmod 3$. By the induction assumption, $T' \in \mathcal{C}$. The tree T can now be constructed from T' by applying Operation **O2**, whence $T \in \mathcal{C}$.

Hence we may assume v is not a remote vertex. Then v lies on the endpaths vxz and vyz'. It follows that $S' = S - \{z'\}$ is a **RDS** for $T' = T - y - z'$. Hence, by reasoning similar to that in the previous paragraph, the tree T can be constructed from T' by applying Operation **O2**, whence $T \in \mathcal{C}$.

Thus, we assume each vertex on a longest path at distance $\text{diam}(T) - 2$ or $\text{diam}(T) - 1$ from r has degree two.

Let v be any vertex on a longest path at distance $\text{diam}(T) - 3$ from r. Let $vx_1y_1z_1$ be an endpath of T. Then $x_1, y_1 \notin S$, and so $v \in S$.

Suppose $\deg(v) \geq 3$. If v is on an endpath vxz, it follows that $x, z \in S$. By the remark following Claim 1, $T \in \mathcal{C}$. Suppose v is a remote vertex adjacent to a leaf u. By Claim 1, u is the only leaf adjacent to v. Moreover, $S' = S - \{u\}$ is a **RDS** for $T' = T - u$. Thus, by Claim 1, $T' \in \mathcal{C}$ and T can be constructed from T' by Operation **O1**, whence $T \in \mathcal{C}$.

So we may assume that v lies only on endpaths $vx_iy_iz_i$, for $i = 1, \ldots, \ell$. Let e be the edge that joins v with its parent, and let $T(v)$ be the component of $T - e$ that contains v. Then $T(v)$ consists of ℓ disjoint paths $x_iy_iz_i$ ($i = 1, \ldots, \ell$) with v joined to x_i for $i = 1, \ldots, \ell$. Let $i \in \{1, \ldots, \ell\}$. Since $x_iy_iz_i$ is an endpath of T, we have $x_i \notin S$, $y_i \notin S$ and $v \in S$. Then $S - \cup_{i=1}^{\ell} \{z_i\}$ is a **RDS** of $T' = T - (T(v) - \{v\})$, and so $\lceil \frac{n-3\ell+2}{3} \rceil \leq \gamma_r(T') \leq \lceil \frac{n+2}{3} \rceil - \ell$, whence $\gamma_r(T') = \lceil \frac{n(T')+2}{3} \rceil$. Thus, $T' \in \mathcal{T}$, and by the induction assumption, $T' \in \mathcal{C}$. Note that v is a leaf of T'. The tree T can now be constructed from T' by applying Operation **O3**, whence $T \in \mathcal{C}$. □

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