Total restrained domination in unicyclic graphs

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Abstract

Let $G = (V, E)$ be a graph. A set $S \subseteq V$ is a total restrained dominating set if every vertex in V is adjacent to a vertex in S and every vertex of $V - S$ is adjacent to a vertex in $V - S$. The total restrained domination number of G, denoted by $\gamma_{tr}(G)$, is the minimum cardinality of a total restrained dominating set of G. A unicyclic graph is a connected graph that contains precisely one cycle. We show that if U is a unicyclic graph of order n, then $\gamma_{tr}(U) \geq \lceil \frac{n}{2} \rceil$, and provide a characterization of graphs achieving this bound.

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1 Introduction

In this paper, we follow the notation of [1]. Specifically, let $G = (V, E)$ be a graph with vertex set V and edge set E. A set $S \subseteq V$ is a *dominating set* (**DS**) of G if every vertex in $V - S$ is adjacent to a vertex in S. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a **DS** of G. The concept of domination in graphs, with its many variations, is now well studied in graph theory. The recent book of Chartrand and Lesniak [1] includes a chapter on domination. A thorough study of domination appears in [6, 7].

In this paper, we continue the study of a variation of the domination theme, namely that of total restrained domination - see [2, 3, 4, 5, 8, 9, 10, 11, 12, 13, 14].

A set $S \subseteq V$ is a *total restrained dominating set* (**TRDS**) if every vertex in V is adjacent to a vertex in S and every vertex in $V - S$ is adjacent to a vertex in $V - S$. Every graph without isolated vertices has a **TRDS**, since $S = V$ is such a set. The *total restrained domination number* of G, denoted by $\gamma_{tr}(G)$, is the minimum cardinality of a **TRDS** of G. A **TRDS** set of G of cardinality $\gamma_{tr}(G)$ is called a γ_{tr}-set of G.

Throughout, n and m denote the order and size of G, respectively. A unicyclic graph U of order n is a connected graph that contains exactly one cycle. Thus, U has size n. A vertex of degree one will be called a *leaf*, while a vertex adjacent to a leaf will be called a *remote vertex*. The *open neighborhood* of a vertex u, denoted $N(u)$, is the set $\{v \in V \,|\, v$ is adjacent to $u\}$, while the *closed neighborhood* of u, denoted $N[u]$, is defined as $N(u) \cup \{u\}$.

A graph G is *status labeled* if every vertex in V is labeled either A or B such that every vertex with label A is adjacent to a vertex with label A and to a vertex with label B, while every vertex with label B is adjacent to a vertex with label B. A vertex $v \in V$ has status A (B, respectively) if v is labeled A (B, respectively). The status of a vertex v will be denoted $\mathrm{Sta}(v)$. We define $\mathrm{Sta}(A)$ ($\mathrm{Sta}(B)$, respectively) as the set of vertices in V with status A (B, respectively).

The following result is due to Cyman and Raczek [3].

Theorem 1 *Let G be a connected graph of order n and size m. Then $\gamma_{tr}(G) \geq \frac{3n}{2} - m$.*

Proof. Let S be a γ_{tr}-set of G and consider $H = \langle V - S \rangle$ and $J = \langle S \rangle$. Let

n_1 and m_1 be the order and size of H, respectively. Moreover, let n_2 and m_2 be the order and size of J, respectively. Thus $m_1 = \frac{1}{2}\sum_{v\in V-S}\deg_H(v) \geq \frac{1}{2}(n-\gamma_{tr}(G))$ and $m_2 = \frac{1}{2}\sum_{v\in S}\deg_J(v) \geq \frac{1}{2}\gamma_{tr}(G)$. Let m_3 denote the number of edges between S and $V-S$. Since S is a **DS**, every vertex in $V-S$ is adjacent to at least one vertex in S. Thus, $m_3 \geq n-\gamma_{tr}(G)$. Hence, $m = m_1+m_2+m_3 \geq \frac{1}{2}(n-\gamma_{tr}(G)) + \frac{1}{2}\gamma_{tr}(G) + n - \gamma_{tr}(G)$, which implies that $\gamma_{tr}(G) \geq \frac{3n}{2} - m$. □

The following known result of [4] is an immediate consequence of Theorem 1.

Corollary 2 *Let T be a tree of order n. Then $\gamma_{tr}(T) \geq \lceil\frac{n+2}{2}\rceil$.*

In similar fashion, we derive our first main result.

Corollary 3 *Let U be a unicyclic graph of order n. Then $\gamma_{tr}(U) \geq \lceil\frac{n}{2}\rceil$.*

Hattingh et al. [4] provided a constructive characterization of trees achieving the lower bound given in Corollary 2, independent of γ_{tr}-set consideration. In the sequel, we constructively characterize unicyclic graphs achieving the lower bound given in Corollary 3, utilizing constructive operations governed by status labeling.

2 Unicylic graphs U of order n with $\gamma_{tr}(U) = \lceil\frac{n}{2}\rceil$

Let $\mathcal{E}$ denote the class of all unicyclic graphs U of order n such that $\gamma_{tr}(U) = \lceil\frac{n}{2}\rceil$. In order to provide the characterization, we state and prove a few observations.

Let $U \in \mathcal{E}$ and let S be a γ_{tr}-set of U.

Observation 1 *If n is even, then every vertex of $V-S$ is adjacent to exactly one vertex of S and adjacent to exactly one vertex of $V-S$, while every vertex in S is adjacent to exactly one vertex of S.*

Proof. Assume n is even and consider the vertex v. If v is a leaf, then $v \in S$. Thus $\deg(v) \geq 2$ for all $v \in V-S$. Now, let $v \in V-S$. Suppose $|N(v)\cap S| \geq 2$. Then $n = m \geq \frac{1}{2}(n-\gamma_{tr}(U)) + \frac{1}{2}\gamma_{tr}(U) + n - \gamma_{tr}(U) +$

1, which implies that $\gamma_{tr}(U) \geq \frac{n+2}{2} > \lceil \frac{n}{2} \rceil$, a contradiction. Suppose $|N(v) \cap (V-S)| \geq 2$. Then $n = m \geq \frac{1}{2}(n - \gamma_{tr}(U) + 1) + \frac{1}{2}\gamma_{tr}(U) + n - \gamma_{tr}(U)$, which implies that $\gamma_{tr}(U) \geq \lceil \frac{n+1}{2} \rceil > \lceil \frac{n}{2} \rceil$, a contradiction. Thus, every vertex in $V - S$ is adjacent to exactly one vertex of S and adjacent to exactly one vertex of $V - S$.

Suppose there is a vertex $y \in S$ such that $|N(y) \cap S| \geq 2$. Then $n = m \geq \frac{1}{2}(n - \gamma_{tr}(G)) + \frac{1}{2}(\gamma_{tr}(G) + 1) + n - \gamma_{tr}(G)$, which implies that $\gamma_{tr}(U) \geq \lceil \frac{n+1}{2} \rceil > \lceil \frac{n}{2} \rceil$, a contradiction. Thus, every vertex in S is adjacent to exactly one vertex of S. $\diamond$

Observation 2 *If n is odd, then S has exactly one of the following properties*:

1. *Every vertex in $V - S$ has degree 2, and there is exactly one vertex $y \in S$ such that $|N(y) \cap S| = 2$, while every other vertex of S is adjacent to exactly one vertex of S.*

2. *There is exactly one vertex $v \in V - S$ such that $\deg(v) = 3$ and $|N(v) \cap (V - S)| = 2$. Furthermore, every vertex in $V - S - \{v\}$ has degree 2, while every vertex in S is adjacent to exactly one vertex of S.*

Proof. Assume n is odd. Let $v \in V - S$ such that $\deg(v) \geq 3$. If $|N(v) \cap S| \geq 2$, then $n = m \geq \frac{1}{2}(n - \gamma_{tr}(U)) + \frac{1}{2}\gamma_{tr}(U) + n - \gamma_{tr}(U) + 1$, and so $\gamma_{tr}(U) \geq \lceil \frac{n+2}{2} \rceil > \lceil \frac{n}{2} \rceil$, a contradiction. Thus, $|N(v) \cap S| = 1$. Suppose $\deg(v) \geq 4$. Then $|N(v) \cap (V - S)| \geq 3$. Thus, $n = m \geq \frac{1}{2}(n - \gamma_{tr}(U) + 2) + \frac{1}{2}\gamma_{tr}(U) + n - \gamma_{tr}(U)$, and so $\gamma_{tr}(U) \geq \lceil \frac{n+2}{2} \rceil > \lceil \frac{n}{2} \rceil$, a contradiction. Hence, $\deg(v) = 3$ and $|N(v) \cap (V - S)| = 2$. Moreover, every vertex in $V - S - \{v\}$ has degree 2. Suppose $y \in S$ such that $|N(y) \cap S| \geq 2$. Then $n = m \geq \frac{1}{2}(n - \gamma_{tr}(U) + 1) + \frac{1}{2}(\gamma_{tr}(U) + 1) + n - \gamma_{tr}(U)$, and so $\gamma_{tr}(U) \geq \lceil \frac{n+2}{2} \rceil > \lceil \frac{n}{2} \rceil$, a contradiction. Thus, $|N(y) \cap S| = 1$ for every vertex $y \in S$, and Property 2 holds.

We may assume that $V - S$ has only degree 2 vertices.

Suppose $|N(y) \cap S| = 1$ for every vertex $y \in S$. Then, both S and $V - S$ induce matchings, and so n is even, which is a contradiction. Thus, there is a vertex $y \in S$ such that $|N(y) \cap S| \geq 2$. Suppose $|N(y) \cap S| \geq 3$. Then $n = m \geq \frac{1}{2}(n - \gamma_{tr}(U)) + \frac{1}{2}\gamma_{tr}(U) + 1 + n - \gamma_{tr}(U)$. Hence, $\gamma_{tr}(U) \geq \lceil \frac{n+2}{2} \rceil > \lceil \frac{n}{2} \rceil$, a contradiction. If $|N(y') \cap S| \geq 2$ for $y' \in S - \{y\}$, we again reach a contradiction. Thus $|N(y) \cap S| = 2$ for exactly one vertex $y \in S$. Therefore, Property 1 holds. $\diamond$

Let P_{ABB} be the status labeled graph obtained from the path P_3 with consecutive vertices p_1, p_2, p_3 by setting $\text{Sta}(p_1) = A$ and $\text{Sta}(p_2) = \text{Sta}(p_3) = B$.

Furthermore, let P_{AABB} be the status labeled graph obtained from the path P_4 with consecutive vertices p_1, p_2, p_3, p_4 by setting $\text{Sta}(p_1) = \text{Sta}(p_2) = A$ and $\text{Sta}(p_3) = \text{Sta}(p_4) = B$. Similarly, let P_{ABBA} be the status labeled graph obtained from the path P_4 with consecutive vertices p_1, p_2, p_3, p_4 by setting $\text{Sta}(p_2) = \text{Sta}(p_3) = B$ and $\text{Sta}(p_1) = \text{Sta}(p_4) = A$. Lastly, let P_{BAAB} be the status labeled graph obtained from the path P_4 with consecutive vertices p_1, p_2, p_3, p_4 by setting $\text{Sta}(p_1) = \text{Sta}(p_4) = B$ and $\text{Sta}(p_2) = \text{Sta}(p_3) = A$.

The following status labeled graphs will serve as the basis for our characterization.

Let B_1 be the status labeled graph obtained from C_4 with consecutive vertices v_1, v_2, v_3, v_4, v_1 by setting $\text{Sta}(v_1) = \text{Sta}(v_2) = B$ and $\text{Sta}(v_3) = \text{Sta}(v_4) = A$.

Let B_2 be the status labeled graph obtained from C_3 with consecutive vertices v_1, v_2, v_3, v_1 by joining v_1 to a vertex w of K_1 and setting $\text{Sta}(v_1) = \text{Sta}(w) = B$ and $\text{Sta}(v_2) = \text{Sta}(v_3) = A$.

Note that if $U \cong B_i$ for $i \in \{1, 2\}$, then $\text{Sta}(B)$ is a γ_{tr}-set of U of cardinality $\lceil \frac{n}{2} \rceil$.

Let U be a status labeled unicyclic graph. Define the following operations on U:

$\mathcal{O}_1$: Suppose v is a vertex of U such that $\text{Sta}(v) = B$. Join v to the vertex p_1 of P_{AABB}.

$\mathcal{O}_2$: Suppose uv is an edge of U. One of the following is performed:

1. If $\text{Sta}(u) = B$ and $\text{Sta}(v) = A$, then delete the edge uv and join u (v, respectively) to vertex p_1 (p_3 or p_4, respectively) of P_{AABB}.
2. If $\text{Sta}(u) = A$ and $\text{Sta}(v) = A$, then delete the edge uv and join u (v, respectively) to vertex p_1 (p_4, respectively) of P_{ABBA}.
3. If $\text{Sta}(u) = B$ and $\text{Sta}(v) = B$, then delete the edge uv and join u (v, respectively) to vertex p_1 (p_4, respectively) of P_{BAAB}.

$\mathcal{O}_3$: Suppose uv is an edge of U, and suppose $\text{Sta}(u) = B$. Delete the edge uv, and join u and v to a vertex w of K_1, setting $\text{Sta}(w) = B$.

$\mathcal{O}_4$: Suppose uv is an edge of U, and suppose $\mathrm{Sta}(u) = \mathrm{Sta}(v) = A$. Delete the edge uv, and join u and v to vertex p_1 of P_{ABB}.

Observation 3 *If U' is the status labeled graph obtained by applying one of the above operations on U, then Sta(B) is a* **TRDS** *of U'.*

Let $\mathcal{C}$ be the family of status labeled graphs U, where U is one of the following four types:

Type 1: U is obtained from B_1 or B_2 by $\ell \geq 0$ applications of $\mathcal{O}_1$ or $\mathcal{O}_2$.

Type 2: U is obtained from:

1. B_1 or B_2 by exactly one application of $\mathcal{O}_4$, followed by $\ell \geq 0$ applications of $\mathcal{O}_1$ or $\mathcal{O}_2$.
2. a **Type 1** graph by joining some $v \in \mathrm{Sta}(A)$ in this **Type 1** graph to vertex p_1 of P_{ABB}, followed by $\ell \geq 0$ applications of $\mathcal{O}_1$ or $\mathcal{O}_2$.

Type 3: U is obtained from a **Type 1** graph by joining some $v \in \mathrm{Sta}(B)$ in this **Type 1** graph to a vertex w of K_1, setting $\mathrm{Sta}(w) = B$, and then following this by $\ell \geq 0$ applications of $\mathcal{O}_1$ or $\mathcal{O}_2$.

Type 4: U obtained from a **Type 1** graph by exactly one application of $\mathcal{O}_3$, followed by $\ell \geq 0$ applications of $\mathcal{O}_1$ or $\mathcal{O}_2$.

Observation 4 *If U is in $\mathcal{C}$, then Sta(B) is a γ_{tr}-set of U of cardinality $\lceil \frac{n}{2} \rceil$.*

Proof. Suppose that U is in $\mathcal{C}$. Then U is of **Type i**, where $1 \leq \mathbf{i} \leq 4$. That $\mathrm{Sta}(B)$ is a **TRDS** of U follows from Observation 3, the fact that if an isolated vertex of status B is joined to any vertex of status B in a status labeled unicyclic graph in which $\mathrm{Sta}(B)$ is a **TRDS**, then in the resulting unicyclic graph $\mathrm{Sta}(B)$ is still a **TRDS**, and the fact that if the vertex p_1 of P_{ABB} is joined to any vertex of status A in a status labeled unicyclic graph in which $\mathrm{Sta}(B)$ is a **TRDS**, then in the resulting unicyclic graph $\mathrm{Sta}(B)$ is still a **TRDS**.

If U is a **Type 1** graph, then $n(U) \equiv 0 \bmod 4$ and $|\mathrm{Sta}(B)| = \frac{n}{2}$, since B_1 or B_2 contribute two vertices out of four to $\mathrm{Sta}(B)$, while each of the $\ell \geq 0$ applications of $\mathcal{O}_1$ or $\mathcal{O}_2$ contributes two vertices out of four to $\mathrm{Sta}(B)$.

Suppose U is a **Type 2** graph, and suppose U is obtained from the **Type 1** graph U' by joining a vertex $v \in \mathrm{Sta}(A)$ in U' to the vertex p_1 of P_{ABB}, and then following this by $\ell \geq 0$ applications of $\mathcal{O}_1$ or $\mathcal{O}_2$.

Then $n(U') \equiv 0 \bmod 4$ and U' has exactly $\frac{n(U')}{2}$ vertices of status B, and so $n(U) \equiv 3 \bmod 4$ and $|\mathrm{Sta}(B)| = \frac{n(U)-3}{2} + 2 = \frac{n+1}{2}$, since P_{ABB} contributes two vertices to $\mathrm{Sta}(B)$ and three to $n(U)$, while each of the applications of $\mathcal{O}_1$ or $\mathcal{O}_2$ contributes two vertices out of four to $\mathrm{Sta}(B)$. As $n \equiv 3 \bmod 4$, we have $\lceil \frac{n}{2} \rceil = \frac{n+1}{2}$, and so $|\mathrm{Sta}(B)| = \lceil \frac{n}{2} \rceil$.

Now, suppose U is obtained from B_1 or B_2 by exactly one application of $\mathcal{O}_4$, followed by $\ell \geq 0$ applications of $\mathcal{O}_1$ or $\mathcal{O}_2$. Again, we have $|\mathrm{Sta}(B)| = \frac{n(U)-3}{2} + 2 = \frac{n+1}{2}$, and as $n \equiv 3 \bmod 4$, we have $\lceil \frac{n}{2} \rceil = \frac{n+1}{2}$, and so $|\mathrm{Sta}(B)| = \lceil \frac{n}{2} \rceil$.

For graphs of **Type 3** and **Type 4**, $n \equiv 1 \bmod 4$, while $|\mathrm{Sta}(B)| = \frac{n-1}{2} + 1 = \frac{n+1}{2} = \lceil \frac{n}{2} \rceil$. $\diamond$

Let U be a unicyclic graph and denote its unique cycle by C. A *reference path* of U is a path $v = u_0, u_1, \ldots, u_t$, where $v \in V(C)$, u_t is a leaf, and $u_i \notin V(C)$ for $i = 1, \ldots, t$. We say a reference path $v = u_0, u_1, \ldots, u_t$ is *maximal* if for every reference path $v = u_0, u_1, u'_2, \ldots, u'_s$ we have that $s \leq t$. We are now ready to state our characterization.

Theorem 4 *Let U be a unicyclic graph of order $n \geq 4$. Then U is in $\mathcal{E}$ if and only if U can be status labeled in such a way that it is in $\mathcal{C}$.*

Proof. Suppose that $U \in \mathcal{C}$. By Observation 4, $U \in \mathcal{E}$.

Now, assume that $U \in \mathcal{E}$ and let S be a γ_{tr}-set of U. We proceed by induction on n. Suppose $n = 4$. If $U = C_4$, then it can be status labeled as B_1 which is in $\mathcal{C}$. If U is the graph obtained from C_3 by joining an isolated vertex to any vertex of C_3, then it can be status labeled as B_2 which is in $\mathcal{C}$. Therefore, assume $n \geq 5$ and, for all $U' \in \mathcal{E}$ such that $4 \leq n(U') < n$, U' can be status labeled so that it is in $\mathcal{C}$. (Henceforth, we will abuse notation slightly by just saying that $U' \in \mathcal{C}$.) Suppose U is a cycle. If n is even, then Observation 1 holds, implying that $n \equiv 0 \bmod 4$. If n is odd, then Property 1 of Observation 2 holds, and so $n \equiv 1 \bmod 4$. Thus U is of **Type 1** or **Type 4**.

Hence, there exists $v \in V(U)$ such that $\deg(v) \geq 3$.

Claim 1 *Suppose $v' = w_0, w_1, \ldots, w_s$ is a maximal reference path of U. If w_{s-1} is adjacent to a vertex $w'_s \in S - \{w_s\}$, then U is of **Type 3**.*

Proof. Note that possibly $w'_s = w_{s-2}$, and that $\{w_{s-1}, w_s, w'_s\} \subseteq S$. By contraposition of Observation 1, $n = 2q+1$, where $q \geq 2$, and Property 1 of Observation 2 holds. Let $U' = U - w_s$, and notice that $S' = S - \{w_s\}$ is a **TRDS** of U', while $n(U') = 2q$. Moreover, S' is a **TRDS** of U' of size $\lceil \frac{2q+1}{2} \rceil - 1 = q$, whence $q = \frac{2q}{2} \leq \gamma_r(U') \leq |S'| = q$. Thus, $U' \in \mathcal{E}$, and by the induction assumption $U' \in \mathcal{C}$. Since $n(U')$ is even, $n(U') \equiv 0 \mod 4$, and so U' is of **Type 1**. Since $w_{s-1} \in \mathrm{Sta}(B)$ in U', U can be obtained from U' by joining w_s to w_{s-1}, and setting $\mathrm{Sta}(w_s) = B$. Hence U is of **Type 3**. $\diamond$

By Claim 1, we conclude that if $v' = w_0, w_1, \ldots, w_s$ is a maximal reference path of U, then $w_{s-1} \in S$ and $\deg(w_{s-1}) = 2$.

Let C denote the unique cycle of U. Among all vertices $v \in C$ such that $\deg(v) \geq 3$, choose the reference path $P : v = u_0, u_1, \ldots, u_t$ for which t is as large as possible. Note that P is necessarily a maximal reference path.

We call a reference path an **R**t path if $\deg(v) = 3$ and $\deg(u_i) = 2$ for $i = 1, \ldots, t-1$. We begin by reducing reference paths to either **R1**, **R2**, **R3** or **R4**.

Case 1. $t \geq 2$.

By Claim 1, $\deg(u_{t-1}) = 2$ and $u_{t-1} \in S$.

We first show that $\deg(u_{t-2}) = 3$ if $t = 2$, while $\deg(u_{t-2}) = 2$ if $t \geq 3$. Suppose, to the contrary, that $\deg(u_{t-2}) \geq 4$ if $t = 2$ and $\deg(u_{t-2}) \geq 3$ if $t \geq 3$. If $u_{t-2} \in S$, then, since $|N(u_{t-2}) \cap S| = |N(u_{t-1}) \cap S| = 2$, Observations 1 and 2 are contradicted. Thus, $u_{t-2} \notin S$ and u_{t-2} is not a remote vertex. But then $|N(u_{t-2}) \cap S| \geq 2$, contradicting Observations 1 and 2.

Thus, if $t = 2$, then $\deg(v) = 3$, while if $t \geq 3$, then $\deg(u_{t-2}) = 2$.

Suppose $t = 3$. Then (cf. Claim 1), we have $u_1 \notin S$, and so $v \notin S$. If $\deg(v) \geq 4$, then Observations 1 and 2 are contradicted. Thus $\deg(v) = 3$.

Suppose $t \geq 4$. We first show that $\deg(u_{t-3}) = 2$. Suppose, to the contrary, that $\deg(u_{t-3}) \geq 3$. Since $u_{t-2} \notin S$, it follows that $u_{t-3} \notin S$. If $\deg(u_{t-3}) \geq 4$, then Observations 1 and 2 are contradicted. Thus $\deg(u_{t-3}) = 3$. By contraposition of Observation 1, $n = 2q+1$ for some positive integer $q \geq 3$, and Property 2 of Observation 2 must hold. Let $U' = U - u_{t-2} - u_{t-1} - u_t$, and notice that $S' = S - \{u_{t-1}, u_t\}$ is a **TRDS** of U'. Thus, U' has order $n - 3 = 2(q-1)$ and $|S'| = q - 1$. Hence, $U' \in \mathcal{E}$, and U' is of **Type 1**. Furthermore, Observation 1 holds for U', and so $\mathrm{Sta}(u_{t-3}) = A$. By joining u_{t-3} to u_{t-2} of $\langle\{u_{t-2}, u_{t-1}, u_t\}\rangle$ in U and setting $\mathrm{Sta}(u_{t-2}) = A$

and $\mathrm{Sta}(u_{t-1}) = \mathrm{Sta}(u_t) = B$, we have that U is of **Type 2**. Therefore, if $t \geq 4$, then $\deg(u_{t-3}) = 2$.

Suppose $t = 4$. We show that $\deg(v) = 3$. Suppose, to the contrary, that $\deg(v) \geq 4$. Since $u_2 \notin S$ and $\deg(u_2) = \deg(u_1) = 2$, it follows that $v \in S$. Suppose v is a remote vertex. Let $U' = U - u_1 - u_2 - u_3 - u_4$, and notice that $S' = S - \{u_3, u_4\}$ is a **TRDS** of U'. Then U' has order $n - 4$, while $|S'| = \lceil \frac{n-4}{2} \rceil$. Thus, $U' \in \mathcal{E}$, and U' is of **Type i**, where $1 \leq \mathbf{i} \leq 4$. Since v is a remote vertex, $\mathrm{Sta}(v) = B$ in U'. By joining v to u_1 of $\langle \{u_1, u_2, u_3, u_4\} \rangle$ in U, and setting $\mathrm{Sta}(u_1) = \mathrm{Sta}(u_2) = A$ and $\mathrm{Sta}(u_3) = \mathrm{Sta}(u_4) = B$, U is of **Type i**, where $1 \leq \mathbf{i} \leq 4$.

Suppose v lies on the maximal reference path v, u_1', u_2'. As $\{v, u_1', u_2'\} \subseteq S$, Claim 1 implies that $U \in C$.

If v lies on the maximal reference path v, u_1', u_2', u_3', then, as $\{v, u_2', u_3'\} \subseteq S$, Observations 1 and 2 are contradicted. Therefore, v lies exclusively on at least two maximal reference paths whose vertices induce P_4. Let $U' = U - u_1 - u_2 - u_3 - u_4$, and notice that $S' = S - \{u_3, u_4\}$ is a **TRDS** of U'. Then U' has order $n - 4$, while $|S'| = \lceil \frac{n-4}{2} \rceil$. Hence, $U' \in \mathcal{E}$ and U' is of **Type i**, where $1 \leq \mathbf{i} \leq 4$. Suppose $\mathrm{Sta}(u_2') = B$. Then Property 1 of Observation 2 must hold, and so $\mathrm{Sta}(u_1') = \mathrm{Sta}(v) = A$, while $\deg_{U'}(u_1') = \deg_{U'}(v) = 2$. But then $\deg_U(v) = 3$, which is a contradiction. Thus, $\mathrm{Sta}(v) = B$. By joining v to u_1 of $\langle \{u_1, u_2, u_3, u_4\} \rangle$ in U, and setting $\mathrm{Sta}(u_1) = \mathrm{Sta}(u_2) = A$ and $\mathrm{Sta}(u_3) = \mathrm{Sta}(u_4) = B$, it follows that U is of **Type i**, where $1 \leq \mathbf{i} \leq 4$. Hence, $\deg(v) = 3$.

Suppose $t \geq 5$. Repeating the arguments above, we may assume $u_{t-4} \in S$ and $\langle \{u_{t-3}, u_{t-2}, u_{t-1}, u_t\} \rangle \cong P_4$. Suppose $\deg(u_{t-4}) \geq 3$. Then u_{t-4} lies exclusively on disjoint paths of the form $u_{t-4}, u_{t-3}^k, u_{t-2}^k, u_{t-1}^k, u_t^k$, where $\langle \{u_{t-3}^k, u_{t-2}^k, u_{t-1}^k, u_t^k\} \rangle \cong P_4$, $u_{t-3}^k \in N(u_{t-4}) - \{u_{t-5}, u_{t-3}\}$ and $1 \leq k \leq |N(u_{t-4})| - 2$. We form U' by removing each u_{t-j} and u_{t-j}^k where $0 \leq j \leq 3$. Then $U' \in \mathcal{E}$, and U' is of **Type i**, where $1 \leq \mathbf{i} \leq 4$. Since u_{t-4} is a leaf of U', it follows that $\mathrm{Sta}(u_{t-4}) = B$ in U'. By re-attaching each path, and labeling the vertices on each path consecutively A, A, B, B, it follows that U is of **Type i**.

Therefore, if $t \geq 4$, then $t = 4$ and $\deg(u_{t-3}) = 2$.

Case 2. $t = 1$. By Claim 1, $\deg(v) = 3$, since otherwise $U \in \mathcal{C}$. Thus, if $t = 1$, then $\deg(v) = 3$.

We have now reduced P to either an **R1**, **R2**, **R3** or **R4** path. We may therefore assume that each reference path of U is either an **R1**, **R2**, **R3** or **R4**.

Suppose that U has an **R2** path v_i, u_1, u_2. We may assume that $v_i \notin S$ and $u_1, u_2 \in S$.

Then $n = 2q + 1$ where $q \geq 1$, and Property 2 of Observation 2 must hold. Thus, $N[v_i] \cap S = \{u_1\}$. If U has a cycle on four, five, or seven vertices, then we are done. If U has a cycle on six vertices, then we reach a contradiction. Thus, U has a cycle on at least eight vertices.

Consider the path $v_{i-2}, v_{i-1}, v_i, v_{i+1}, \ldots, v_{i+5}$ on C, where $v_{i-1}, v_i, v_{i+1} \notin S$ and $v_{i+2}, v_{i-2} \in S$. By symmetry, without loss of generality, suppose v_{i-2} lies on an **R4** or **R1** path.

First consider the case when $v_{i+3} \in S$. Then, by Property 2 of Observation 2, it follows that $v_{i+4} \notin S$, while neither v_{i+2} nor v_{i+3} are on any **Ri** paths for $1 \leq i \leq 3$. Thus $v_{i+5} \notin S$. Let r $(0 \leq r \leq 2)$ be the number of **R4** paths originating from v_{i+2} and v_{i+3}. We form U' by removing $v_{i+1}, v_{i+2}, v_{i+3}, v_{i+4}$, and the $4r$ vertices of the **R4** paths, and then joining v_i and v_{i+5}. Then U has order $n - 4 - 4r = 2(q - 2r - 2) + 1$, and $\gamma_{tr}(U') = q - 2r - 1$. Thus, $U' \in \mathcal{E}$ and U' is of **Type i**, where $\mathbf{i} \in \{2, 3, 4\}$. Moreover, Observation 2 holds, and so $\text{Sta}(v_{i-2}) = B$ (since otherwise v_{i-2} is on the **R4** path $v_{i-2}, u_1', u_2', u_3', u_4'$ where $\{u_2', u_3', u_4'\} \subseteq \text{Sta}(B)$, which contradicts Property 2 of Observation 2). If $\text{Sta}(v_i) = B$, then $\{v_{i-1}, v_i, u_1, u_2\} \subseteq \text{Sta}(B)$, contradicting Observation 2. Thus, $\text{Sta}(v_i) = A$, and Property 2 of Observation 2 holds, and so $\text{Sta}(v_{i+5}) = \text{Sta}(v_{i-1}) = A$. Delete the edge $v_i v_{i+5}$, join v_i (v_{i+5}, respectively) to v_{i+1} (v_{i+4}, respectively) of $\langle \{v_{i+1}, v_{i+2}, v_{i+3}, v_{i+4}\} \rangle$ in U, and set $\text{Sta}(v_{i+1}) = \text{Sta}(v_{i+4}) = A$ and $\text{Sta}(v_{i+2}) = \text{Sta}(v_{i+3}) = B$. By applying $\mathcal{O}_1$ (if necessary) to v_{i+2} and v_{i+3}, we obtain U. Thus, U is of **Type i**, where $\mathbf{i} \in \{2, 3, 4\}$.

Next consider the case when $v_{i+3} \notin S$. Then v_{i+2} must lie on an **R1** path v_{i+2}, u_1'. Furthermore, $v_{i+3}, v_{i+4} \notin S$, and $v_{i+5} \in S$. We form U' by removing $v_{i+2}, v_{i+3}, v_{i+4}, u_1'$, and then joining v_{i+1} and v_{i+5}. Then U' has order $n - 4 = 2(q - 2) + 1$ and $\gamma_{tr}(U') = q - 1$. Thus, $U' \in \mathcal{E}$ and U' is of **Type i**, where $\mathbf{i} \in \{2, 3, 4\}$. Moreover, Observation 2 holds, and $\text{Sta}(v_{i-2}) = B$. If $\text{Sta}(v_i) = B$, then $\{v_i, v_{i-1}, u_1, u_2\} \subseteq \text{Sta}(B)$, contradicting Observation 2. Thus, $\text{Sta}(v_i) = A$, and by Property 2 of Observation 2, $\text{Sta}(v_{i+1}) = \text{Sta}(v_{i-1}) = A$. Furthermore, $\text{Sta}(v_{i+5}) = B$. Delete the edge $v_{i+1} v_{i+5}$, and join v_{i+1} (v_{i+5}, respectively) to v_{i+2} (v_{i+4}, respectively) of $\langle \{v_{i+2}, v_{i+3}, v_{i+4}, u_1'\} \rangle$ in U, and set $\text{Sta}(v_{i+3}) = \text{Sta}(v_{i+4}) = A$ and $\text{Sta}(v_{i+2}) = \text{Sta}(u_1') = B$. Thus, U is of **Type i**, where $\mathbf{i} \in \{2, 3, 4\}$.

We may assume that neither v_{i-2} nor v_{i+2} lies on an **R1** or **R4** path. Then the cycle of U has length as least nine. We now consider the path $v_{i-1}, v_i, v_{i+1}, \ldots, v_{i+6}$, where $v_{i+2}, v_{i+3} \in S$ and $v_{i+4}, v_{i+5} \notin S$. Let r

$(0 \leq r \leq 1)$ be the number of **R4** paths on v_{i+3}. We form U' by removing $v_{i+2}, v_{i+3}, v_{i+4}, v_{i+5}$, and the $4r$ vertices of the possible **R4** paths, and then joining v_{i+1} and v_{i+6}. Then U' has order $n - 4 - 4r = 2(q - 2r - 2) + 1$, and $\gamma_{tr}(U') = q - 2r - 1$. Thus, $U' \in \mathcal{E}$, and U' is of **Type i**, where $\mathbf{i} \in \{2, 3, 4\}$. Moreover, Observation 2 holds. Suppose $\text{Sta}(v_i) = B$. By Observation 2, $\text{Sta}(v_{i+1}) = \text{Sta}(v_{i-1}) = A$, and so $\text{Sta}(v_{i+6}) = A$. Delete the edge $v_{i+1}v_{i+6}$, and join v_{i+1} (v_{i+6}, respectively) to v_{i+2} (v_{i+5}, respectively) in $\langle\{v_{i+2}, v_{i+3}, v_{i+4}, v_{i+5}\}\rangle$, and set $\text{Sta}(v_{i+2}) = \text{Sta}(v_{i+5}) = A$ and $\text{Sta}(v_{i+3}) = \text{Sta}(v_{i+4}) = B$. By applying $\mathcal{O}_1$ (if necessary) to v_{i+3}, it follows that U is of **Type i**, where $\mathbf{i} \in \{2, 3, 4\}$. Suppose $\text{Sta}(v_i) = A$. By Observation 2, $\text{Sta}(v_{i-1}) = \text{Sta}(v_{i+1}) = A$. Furthermore, $\text{Sta}(v_{i+6}) = B$. Delete the edge $v_{i+1}v_{i+6}$, and join v_{i+1} (v_{i+6}, respectively) to v_{i+2} (v_{i+5}, respectively) in $\langle\{v_{i+2}, v_{i+3}, v_{i+4}, v_{i+5}\}\rangle$, and set $\text{Sta}(v_{i+4}) = \text{Sta}(v_{i+5}) = A$ and $\text{Sta}(v_{i+2}) = \text{Sta}(v_{i+3}) = B$. By applying $\mathcal{O}_1$ (if necessary) to v_{i+3}, it follows that U is of **Type i**, where $\mathbf{i} \in \{2, 3, 4\}$.

We may assume that U has no **R2** paths.

Suppose that U has an **R1** path v_i, u_1. If U has a cycle on three or four vertices, then we are done. Thus, U has a cycle on more than four vertices. Let v_{i-1} and v_{i+1} be neighbors of v_i that lie C. If $\{v_{i-1}, v_{i+1}\} \subseteq S$, then $|N(v_i) \cap S| = 3$, contradicting Observations 1 and 2. Without loss of generality, suppose that $v_{i+1} \notin S$. If U has a cycle on five vertices, then we reach a contradiction. If U has a cycle on six vertices, then we are done. Thus, U has a cycle on at least seven vertices. Consider the path $v_i, v_{i+1}, \ldots, v_{i+6}$, on C. If $v_{i+2} \in S$, then $\deg(v_{i+1}) = 3$ and $|N(v_{i+1}) \cap S| = 2$, contradicting Observations 1 and 2. Thus, $v_{i+2} \notin S$, and since U has no **R2** paths, $v_{i+3} \in S$.

Case 2.1 $v_{i+4} \in S$.

Case 2.1.1 $v_{i+5} \in S$.

By contraposition of Observation 1, $n = 2q + 1$ where $q \geq 3$. Moreover, as Property 1 of Observation 2 holds, $v_{i+6} \notin S$, $\deg(v_{i+1}) = \deg(v_{i+2}) = 2$, while v_{i+3}, v_{i+4} and v_{i+5} do not lie on either an **R1** path or an **R3** path. Let r $(0 \leq r \leq 3)$ be the number of **R4** paths on v_{i+3}, v_{i+4} and v_{i+5}. We form U' by removing $v_{i+1}, v_{i+2}, v_{i+3}, v_{i+4}$ and v_{i+5}, and the $4r$ vertices of the aforementioned **R4** paths, and then joining v_i to v_{i+6}. Then U' has order $n - 5 - 4r = 2(q - 2r - 2)$ and $\gamma_{tr}(U') = q - 2r - 2$. Hence, $U' \in \mathcal{E}$ and U' is of **Type 1**. By Observation 1, $\text{Sta}(v_i) = B$, and so $\text{Sta}(v_{i+6}) = A$. Delete the edge v_iv_{i+6}, and join v_i (v_{i+6}, respectively) to vertex v_{i+1} (v_{i+5}, respectively) of the path P_4 with consecutive vertices $v_{i+1}, v_{i+2}, v_{i+3}, v_{i+5}$, and set $\text{Sta}(v_{i+1}) = \text{Sta}(v_{i+2}) = A$ and $\text{Sta}(v_{i+3}) = \text{Sta}(v_{i+5}) = B$. Delete the edge $v_{i+3}v_{i+5}$, and join v_{i+3} and v_{i+5} to v_{i+4}, and set $\text{Sta}(v_{i+4}) = B$. By applying $\mathcal{O}_1$ (if necessary) to v_{i+3}, v_{i+4} and v_{i+5}, it follows that U is of **Type 4**.

Case 2.1.2 $v_{i+5} \notin S$.

At most one of v_{i+1} and v_{i+2} can lie on an **R3** path. Without loss of generality, suppose that v_{i+1} lies on an **R3** path. Thus, $n = 2q + 1$ where $q \geq 3$ and Property 2 of Observation 2 holds. Moreover, neither v_{i+3} nor v_{i+4} lies on an **R1** path. Let r $(0 \leq r \leq 2)$ be the number of **R4** paths on v_{i+3} and v_{i+4}. We form U' by removing $v_{i+1}, v_{i+2}, v_{i+3}$ and v_{i+4}, the $4r$ vertices of the possible **R4** paths, and the three vertices from the **R3** path, and then joining v_i to v_{i+5}. Then U' has order $n-4r-7 = 2(q-2r-3)$ and $\gamma_{tr}(U') = q - 2r - 3$. Thus $U' \in \mathcal{E}$ and U' is of **Type 1**. As Observation 1 holds, $\text{Sta}(v_i) = B$, and so $\text{Sta}(v_{i+5}) = A$. Delete the edge v_iv_{i+5} and join v_i (v_{i+5}, respectively) to v_{i+1} (v_{i+4}, respectively) in $\langle\{v_{i+1}, v_{i+2}, v_{i+3}, v_{i+4}\}\rangle$ of U, and set $\text{Sta}(v_{i+1}) = \text{Sta}(v_{i+2}) = A$ and $\text{Sta}(v_{i+3})$ $=\text{Sta}(v_{i+4}) = B$. Then, join v_{i+1} to vertex p_1 of P_{ABB}. By applying $\mathcal{O}_1$ (if necessary) to v_{i+3} and v_{i+4}, it follows that U is of **Type 2**.

Thus, v_{i+1} and v_{i+2} do not lie on an **R3** path, and therefore $\deg(v_{i+1}) = \deg(v_{i+2}) = 2$. At most one of v_{i+3} and v_{i+4} lies on an **R1** path. Without loss of generality, suppose that v_{i+3} lies on an **R1** path. Then $n = 2q + 1$ where $q \geq 3$, and Property 1 of Observation 2 holds. Let r $(0 \leq r \leq 1)$ denote the number of **R4** paths on v_{i+4}. We form U by removing $v_{i+1}, v_{i+2}, v_{i+3}$ and v_{i+4}, the $4r$ vertices of the possible **R4** paths, and the one vertex from the **R1** path, and then joining v_i to v_{i+5}. Then U has order $n - 4r - 5 = 2(q - 2r - 2)$ and so $\gamma_{tr}(U') = q - 2r - 2$, whence $U' \in \mathcal{E}$. Furthermore, U' must be of **Type 1**. As Observation 1 holds, $\text{Sta}(v_i) = B$, and so $\text{Sta}(v_{i+5}) = A$. Delete the edge v_iv_{i+5} and join v_i (v_{i+5}, respectively) to v_{i+1} (v_{i+4}, respectively) in $\langle\{v_{i+1}, v_{i+2}, v_{i+3}, v_{i+4}\}\rangle$, and set $\text{Sta}(v_{i+1}) = \text{Sta}(v_{i+2}) = A$ and $\text{Sta}(v_{i+3})$ $=\text{Sta}(v_{i+4}) = B$. Then join v_{i+3} to a vertex w of K_1 and set $\text{Sta}(w) = B$. By applying $\mathcal{O}_1$ (if necessary) to v_{i+4}, it follows that U is of **Type 3**.

Suppose that neither v_{i+3} nor v_{i+4} lies on an **R1** path. Let r $(0 \leq r \leq 2)$ denote the number of **R4** paths on v_{i+3} and v_{i+4}. We form U' by removing $v_{i+1}, v_{i+2}, v_{i+3}$ and v_{i+4}, and the $4r$ vertices of the possible **R4** paths, and then joining v_i to v_{i+5}. Again, $U' \in \mathcal{E}$, and U' is of **Type i**, where $1 \leq \mathbf{i} \leq 4$. Notice that $\text{Sta}(v_i) = B$. Suppose $\text{Sta}(v_{i+5}) = B$. Delete the edge v_iv_{i+5} and join v_i (v_{i+5}, respectively) to v_{i+1} (v_{i+4}, respectively) in $\langle\{v_{i+1}, v_{i+2}, v_{i+3}, v_{i+4}\}\rangle$, and set $\text{Sta}(v_{i+1}) = \text{Sta}(v_{i+2}) = A$ and $\text{Sta}(v_{i+3})$ $=\text{Sta}(v_{i+4}) = B$. By applying $\mathcal{O}_1$ (if necessary) on v_{i+3} and v_{i+4}, it follows that $\text{Sta}(B)$ is a γ_{tr}-set of U that contains v_{i+5}, and so we have **Case 1.1** above. We may assume that $\mathrm{Sta}(v_{i+5}) = A$. Delete the edge v_iv_{i+5} and join v_i (v_{i+5}, respectively) to v_{i+1} (v_{i+4}, respectively) in $\langle\{v_{i+1}, v_{i+2}, v_{i+3}, v_{i+4}\}\rangle$ of U, and set $\mathrm{Sta}(v_{i+1}) = \mathrm{Sta}(v_{i+2}) = A$ and $\mathrm{Sta}(v_{i+3})$ =$\mathrm{Sta}(v_{i+4}) = B$. By applying $\mathcal{O}_1$ (if necessary) to v_{i+3} and v_{i+4}, it follows that U is of **Type i**.

Case 2.2 $v_{i+4} \notin S$.

Then v_{i+3} lies on an **R1** path v_{i+3}, u'_1. Assume first that v_{i+1} and v_{i+2} do not lie on an **R3** path. We form U' by removing $v_{i+1}, v_{i+2}, v_{i+3}$, and u'_1, and then joining v_i and v_{i+4}. Then $U' \in \mathcal{E}$ and U' is of **Type i**, where $1 \leq \mathbf{i} \leq 4$. Notice that $\mathrm{Sta}(v_i) = B$. Suppose $\mathrm{Sta}(v_{i+4}) = B$. Delete the edge v_iv_{i+4} and join v_i (v_{i+4}, respectively) to v_{i+1} (v_{i+3}, respectively) in $\langle\{v_{i+1}, v_{i+2}, v_{i+3}, u'_1\}\rangle$, and set $\mathrm{Sta}(v_{i+1}) = \mathrm{Sta}(v_{i+2}) = A$ and $\mathrm{Sta}(v_{i+3})$ =$\mathrm{Sta}(u'_1) = B$. It follows that $\mathrm{Sta}(B)$ is a γ_{tr}-set of U that contains v_{i+4}, and so we have **Case 1** of this proof. Thus, $\mathrm{Sta}(v_{i+4}) = A$. By similar reasoning to that above, it follows that U is of **Type i**.

Now, at most one of v_{i+1} and v_{i+2} lies on an **R3** path. Without loss of generality, suppose that v_{i+1} lies on an **R3** path. Then $n = 2q + 1$ where $q \geq 2$, and Property 2 of Observation 2 holds. We form U' by removing $v_{i+1}, v_{i+2}, v_{i+3}, u'_1$, and the three vertices from the **R3** path, and then joining v_i and v_{i+4}. Then U' has order $n-7 = 2(q-3)$, and $\gamma_{tr}(U') = q - 3$. Thus $U' \in \mathcal{E}$ and U' must be of **Type 1**. By Observation 1, $\mathrm{Sta}(v_i) = B$, and so $\mathrm{Sta}(v_{i+4}) = A$. Delete the edge v_iv_{i+4} and join v_i (v_{i+4}, respectively) to v_{i+1} (v_{i+3}, respectively) in $\langle\{v_{i+1}, v_{i+2}, v_{i+3}, u'_1\}\rangle$, and set $\mathrm{Sta}(v_{i+1}) = \mathrm{Sta}(v_{i+2}) = A$ and $\mathrm{Sta}(v_{i+3}) = \mathrm{Sta}(u'_1) = B$. Then join v_{i+1} to vertex p_1 of P_{ABB}. It follows that U is of **Type 2**.

We may assume that U has no **R1** paths.

Assume that U has an **R3** path v, u_1, u_2, u_3. Since $v \notin S$, Property 2 of Observation 2 holds. Moreover, U has exactly one **R3** path, $u_1 \notin S$, and $u_2, u_3 \in S$. Let $C = v, v_1, \ldots, v_{N-1}, v_N, v$ denote the cycle of U, and notice that $n(\langle C\rangle) \equiv 0 \bmod 4$. Without loss of generality suppose $v_N \in S$, and notice that $v_i \in S$ for each and only each $i \equiv 2$ or $3 \bmod 4$ $(1 \leq i \leq N)$.

Let r be the number of **R4** paths on vertices of C, which emanate only from vertices on $C \cap S$. We form U' by removing the $4r$ vertices of the aforementioned **R4** paths, $v_1, \ldots, v_{N-3}$ (if $N \geq 7$) and u_1, u_2, u_3, and then joining v to v_{N-2}. By setting $\mathrm{Sta}(v) = \mathrm{Sta}(v_{N-2}) = A$ and $\mathrm{Sta}(v_N) = \mathrm{Sta}(v_{N-1}) = B$, it follows that $U' \cong B_1$, whence U' is of **Type 1**. Join v to u_1 of $\langle\{u_1, u_2, u_3\}\rangle$, setting $\mathrm{Sta}(u_1) = A$ and $\mathrm{Sta}(u_2) = \mathrm{Sta}(u_3) = B$, and so the resulting graph is of **Type 2**. Then delete the edge vv_{N-2} and re-insert $v_1, \ldots, v_{N-3}$ (by applying $\mathcal{O}_2$ zero or more times), setting $\mathrm{Sta}(v_i) = A$ for

$i \equiv 0$ or 1 mod 4, and $\mathrm{Sta}(v_i) = B$ for $i \equiv 2$ or 3 mod 4. Finally, by re-attaching the **R4** paths with the natural labeling A, A, B, B, it follows that U is of **Type 2**.

Thus, assume that U has only **R4** paths. These paths emanate only from vertices on $C \cap S$, where $C = v, v_1, \ldots, v_{N-1}, v_N, v$ denote the cycle of U. Note that $n(\langle C \rangle) \equiv 0$ or 1 mod 4. Without loss of generality, suppose $v_{N-1}, v_N \in S$.

First, suppose that $n(\langle C \rangle) \equiv 0$ mod 4. Then $v \notin S$, and $v_i \in S$ for each and only each $i \equiv 2$ or 3 mod 4 ($1 \leq i \leq N$). Let r be the number of **R4** paths on vertices of C. We form U' by removing the $4r$ vertices of the aforementioned **R4** paths, and $v_1, \ldots, v_{N-3}$ (if $N \geq 7$), and then joining v to v_{N-2}. By setting $\mathrm{Sta}(v) = \mathrm{Sta}(v_{N-2}) = A$ and $\mathrm{Sta}(v_N) = \mathrm{Sta}(v_{N-1}) = B$, it follows that $U' \cong B_1$, whence U' is of **Type 1**. Delete the edge vv_{N-2} and re-insert $v_1, \ldots, v_{N-3}$ (by applying $\mathcal{O}_2$ zero or more times), setting $\mathrm{Sta}(v_i) = A$ for $i \equiv 0$ or 1 mod 4, and $\mathrm{Sta}(v_i) = B$ for $i \equiv 2$ or 3 mod 4. Finally, by re-attaching the **R4** paths with the natural labeling, it follows that U is of **Type 1**.

Now consider the case when $n(\langle C \rangle) \equiv 1$ mod 4. Since U has only **R4** paths, all degree three vertices of U are in S. Thus Property 1 of Observation 2 holds. Without loss of generality suppose $v_{N-2} \in S$. Then $v \notin S$, and $v_i \in S$ for each and only each $i \equiv 2$ or 3 mod 4 ($1 \leq i \leq N$). Let r be the number of **R4** paths on vertices of C. We form U' by removing the $4r$ vertices of the aforementioned **R4** paths, $v_1, \ldots, v_{i-4}$ (if $N \geq 8$), the vertex v_{N-1}, and then joining v to v_{N-3} and v_{N-2} to v_N. By setting $\mathrm{Sta}(v) = \mathrm{Sta}(v_{N-3}) = A$ and $\mathrm{Sta}(v_N) = \mathrm{Sta}(v_{N-2}) = B$, it follows that $U' \cong B_1$, whence U' is of **Type 1**. Delete the edge $v_{N-2}v_N$, and join the vertex v_{N-1} to the vertices v_{N-2} and v_N. By setting $\mathrm{Sta}(v_{N-1}) = B$, the resulting graph is of **Type 4**. Delete the edge vv_{N-3} and re-insert $v_1, \ldots, v_{N-4}$ (by applying $\mathcal{O}_2$ zero or more times), setting $\mathrm{Sta}(v_i) = A$ for $i \equiv 0$ or 1 mod 4, and $\mathrm{Sta}(v_i) = B$ for $i \equiv 2$ or 3 mod 4. Finally, by re-attaching the **R4** paths with the natural labeling, it follows that U is of **Type 4**. As we have shown that $U \in \mathcal{C}$, the proof is complete. □

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