Restrained Bondage in Graphs

Johannes H. Hattingh and Andrew R. Plummer

Department of Mathematics and Statistics
University Plaza
Georgia State University
Atlanta, Georgia 30303, USA

Abstract

Let $G = (V, E)$ be a graph. A set $S \subseteq V$ is a restrained dominating set if every vertex not in S is adjacent to a vertex in S and to a vertex in $V - S$. The restrained domination number of G, denoted by $\gamma_r(G)$, is the smallest cardinality of a restrained dominating set of G. We define the *restrained bondage number* $b_r(G)$ of a nonempty graph G to be the minimum cardinality among all sets of edges $E' \subseteq E$ for which $\gamma_r(G - E') > \gamma_r(G)$. Sharp bounds are obtained for $b_r(G)$, and exact values are determined for several classes of graphs. Also, we show that the decision problem for $b_r(G)$ is NP-complete even for bipartite graphs.

Keywords: Restrained domination; Bondage number; Restrained bondage
MSC: 05C69

1 Introduction

In this paper, we follow the notation of [2]. Specifically, let $G = (V, E)$ be a graph with vertex set V and edge set E. Moreover, the notation P_n will denote the path of order n, and the notation S_n will denote the star graph of order n. A set $S \subseteq V$ is a *dominating set* of G if every vertex not in S is adjacent to a vertex in S. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G. The concept of domination in graphs, with its many variations, is now well studied in graph theory. The recent book of Chartrand and Lesniak [2] includes a chapter on domination. A thorough study of domination appears in [9, 10].

In this paper, we continue the study of a variation of the domination theme, namely that of restrained domination [3, 4, 5, 11, 12]. A set $S \subseteq V$ is a *restrained dominating set* (**RDS**) if every vertex not in S is adjacent to a vertex in S and to a vertex in $V - S$. Every graph has a **RDS**, since $S = V$ is such a set. The *restrained domination number* of G, denoted by $\gamma_r(G)$, is the minimum cardinality of a **RDS** of G. A **RDS** S is called a $\gamma_r(G)$-set of G if $|S| = \gamma_r(G)$.

The concept of restrained domination was introduced by Telle and Proskurowski [12], albeit indirectly, as a vertex partitioning problem. Here conditions are imposed on a set S, the complementary set $V-S$ and on edges between the sets S and $V-S$. For example, if we require that every vertex in $V-S$ should be adjacent to some other vertex of $V-S$ (the condition on the set $V-S$) and to some vertex in S (the condition on edges between the sets S and $V-S$), then S is a **RDS**.

One application of domination is that of prisoners and guards. For security, each prisoner must be seen by some guard; the concept is that of domination. However, in order to protect the rights of prisoners, we may also require that each prisoner is seen by another prisoner; the concept is that of restrained domination.

The *bondage number* $b(G)$ of a nonempty graph G is the minimum cardinality among all sets of edges $E' \subseteq E$ for which $\gamma(G-E') > \gamma(G)$. Thus, the bondage number of G is the smallest number of edges whose removal renders every minimum dominating set of G a "nondominating" set in the resultant spanning subgraph. Since the domination number of every spanning subgraph of a nonempty graph G is at least as great as $\gamma(G)$, the bondage number of a nonempty graph is well defined. This concept was introduced by Bauer, Harary, Nieminen and Suffel [1] and has been further studied by Fink, Jacobson, Kinch and Roberts [6], Hartnell and Rall [8] and Teschner[13].

Herein we further the study of bondage by considering a variation based on restrained domination. Ergo, the *restrained bondage number* $b_r(G)$ of a nonempty graph G is the minimum cardinality among all sets of edges $E' \subseteq E$ for which $\gamma_r(G-E') > \gamma_r(G)$. Thus, the restrained bondage number of G is the smallest number of edges whose removal renders every $\gamma_r(G)$-set of G either a "nondominating" set or an "unrestrained" set in the resultant spanning subgraph.

We define a *remote vertex* as a vertex adjacent to a leaf. A *galaxy* is a forest in which each component is a star. Note that when G is a galaxy, there exists no set of edges E' such that $\gamma_r(G-E') > \gamma_r(G)$. Conversely, suppose there exists no set of edges E' of G such that $\gamma_r(G-E') > \gamma_r(G)$. We show that G is a galaxy. For suppose, to the contrary, that component K is not a star. Then K either contains a C_3 with vertex set $\{u_1, u_2, u_3\}$ or a P_4 with vertex set $\{u_1, u_2, u_3, u_4\}$. But then $\gamma_r(G) \leq |V(G) - \{u_2, u_3\}| = n-2$, while $\gamma_r(G-E') = n$ where $E' = E(G)$. Thus, $\gamma_r(G-E') > \gamma_r(G)$ for the set of edges $E(G)$, which is a contradiction. Thus, there exists a set of edges E' such that $\gamma_r(G-E') > \gamma_r(G)$ if and only if G is not a galaxy. The restrained bondage number of a graph G is therefore only defined for a graph G which is not a galaxy.

2 Exact values for $b_r(G)$

Proposition 1 *For the complete graph K_n $(n \geq 3)$,*

$$b_r(K_n) = \begin{cases} 1 & \text{if } n = 3 \\ \lceil \frac{n}{2} \rceil & \text{otherwise.} \end{cases}$$

Proof. Assume $n = 3$. Clearly $\gamma_r(K_3) = 1$. Now, removing any edge from K_3 yields P_3. Since $\gamma_r(P_3) = 3$, it follows that $b_r(K_3) = 1$. Let $n \geq 4$ and let H be a spanning subgraph of K_n that is obtained by removing fewer than $\lceil \frac{n}{2} \rceil$ edges from K_n. Then H contains a vertex of degree $n-1$. Moreover, for every $v \in V(H)$, $\deg_H(v) \geq 2$. Hence, $\gamma_r(H) = 1$. It follows that $b_r(K_n) \geq \lceil \frac{n}{2} \rceil$.

Assume n is even. Let H be the graph obtained by removing $n/2$ independent edges from K_n. Thus, for every $v \in V(H)$, $\deg_H(v) = n-2$, whence $\gamma_r(H) = 2$. Assume n is odd and let H' be the graph obtained by removing $(n-1)/2$ independent edges from K_n. Thus, there is exactly one vertex $v \in V(H')$ such that $\deg_{H'}(v) = n-1$. Let H be the graph obtained by removing from H' one edge incident with v. It follows that $\gamma_r(H) = 2$. In either case, H results from the removal of $\lceil \frac{n}{2} \rceil$ edges from K_n. Thus $b_r(K_n) \leq \lceil \frac{n}{2} \rceil$, whence $b_r(K_n) = \lceil \frac{n}{2} \rceil$. □

Proposition 2 [5] *If $n \geq 3$, then $\gamma_r(C_n) = n - 2\lfloor \frac{n}{3} \rfloor$. Moreover, if $n \geq 1$, then $\gamma_r(P_n) = n - 2\lfloor \frac{n-1}{3} \rfloor$.*

Corollary 3 *If $n \geq 3$, then*

$$\gamma_r(C_n) = \begin{cases} \lceil \frac{n}{3} \rceil + 1 & \text{if } n \equiv 2 \bmod 3 \\ \lceil \frac{n}{3} \rceil & \text{otherwise.} \end{cases}$$

Moreover, if $n \geq 1$, then

$$\gamma_r(P_n) = \begin{cases} \lceil \frac{n}{3} \rceil + 2 & \text{if } n \equiv 0 \bmod 3 \\ \lceil \frac{n}{3} \rceil + 1 & \text{if } n \equiv 2 \bmod 3 \\ \lceil \frac{n}{3} \rceil & \text{otherwise.} \end{cases}$$

Proposition 4 *If $n \geq 3$, then*

$$b_r(C_n) = \begin{cases} 1 & \text{if } n \equiv 0 \bmod 3 \\ 2 & \text{otherwise.} \end{cases}$$

Proof. Assume $n \equiv 0 \bmod 3$. Since $\gamma_r(C_n) < \gamma_r(P_n)$, $b_r(C_n) = 1$.

Thus, assume $n \equiv i \bmod 3$ $(i = 1, 2)$. Since $\gamma_r(C_n) = \gamma_r(P_n)$, it follows that $b_r(C_n) \geq 2$. Let H be the graph obtained by the removal of two edges from C_n such that P_3 and P_{n-3} are formed. Then $\gamma_r(H) = \gamma_r(P_{n-3}) + \gamma_r(P_3) = (\lceil \frac{n-3}{3} \rceil + i - 1) + 3 = (\lceil \frac{n}{3} - 1 \rceil + i - 1) + 3 = (\lceil \frac{n}{3} \rceil + i - 1) + 2 = \gamma_r(C_n) + 2 > \gamma_r(C_n)$.

Thus, $\gamma_r(H) > \gamma_r(C_n)$, whence $b_r(C_n) \leq 2$. Hence, $b_r(C_n) = 2$. □

Theorem 5 *If P_n is a path of order $n \geq 4$, then $b_r(P_n) = 1$.*

Proof. Assume $n \equiv i \bmod 3$ $(i = 1, 2)$. Since $\gamma_r(P_n) = \gamma_r(C_n)$, by reasoning similar to that in the previous proof, we have $b_r(P_n) \leq 1$, whence $b_r(P_n) = 1$. Assume $n \equiv 0 \bmod 3$. Let H be the graph obtained by the removal of one edge from P_n such that P_3 and P_{n-3} are formed. Then $\gamma_r(H) = \gamma_r(P_{n-3}) + \gamma_r(P_3) = (\lceil \frac{n-3}{3} \rceil + 2) + 3 = (\lceil \frac{n}{3} - 1 \rceil + 2) + 3 = \lceil \frac{n}{3} \rceil - 1 + 2 + 3 = (\lceil \frac{n}{3} \rceil + 2) + 2 = \gamma_r(P_n) + 2 > \gamma_r(P_n)$.

Thus, $\gamma_r(H) > \gamma_r(P_n)$, whence $b_r(P_n) \leq 1$. Hence, $b_r(P_n) = 1$. $\square$

Theorem 6 *Let T be a tree of order $n \geq 4$. Then $T \ncong S_n$ if and only if $b_r(T) = 1$.*

Proof. Since $n \geq 4$ and $T \ncong S_n$, it follows that $\mathrm{diam}(T) \geq 3$. Assume $\mathrm{diam}(T) = 3$. Then T is a double star. Let $L(T)$ denote the set of leaves of T, and notice that $L(T)$ is the unique $\gamma_r(T)$-set of T. Hence, $\gamma_r(T) = n - 2$. Let $a, b \in V(T) - L(T)$, and consider $T' = T - ab$. Since T' comprises two stars, it follows immediately that $\gamma_r(T') = n$, and so $b_r(T) = 1$. Therefore, assume that $\mathrm{diam}(T) \geq 4$. Suppose to the contrary that $b_r(T) \geq 2$. Let T be rooted at a leaf r of a longest path. Let v be any vertex on a longest path P at distance $\mathrm{diam}(T) - 1$ from r. Let w be the vertex on P at distance $\mathrm{diam}(T) - 2$ from r adjacent to v, and let x be the vertex on P at distance $\mathrm{diam}(T) - 3$ from r adjacent to w.

Suppose $\deg(w) = 2$ and consider $T' = T - xw$. Let T'_x denote the component of T' containing x and let T'_w denote the component of T' containing w. Since $b_r(T) \geq 2$, it follows that $\gamma_r(T) = \gamma_r(T') = \gamma_r(T'_x) + \gamma_r(T'_w)$. Moreover, since $\deg(w) = 2$, it follows that $T'_w \cong S_k$, where $k = |V(T'_w)|$. Therefore, $\gamma_r(T'_w) = k$, and $\gamma_r(T') = \gamma_r(T'_x) + k$. Let R' be a $\gamma_r(T')$-set of T', and notice that $V(T'_w) \subseteq R'$. If $x \in R'$, then $R' - \{w, v\}$ is an **RDS** of T, and if $x \notin R'$, then $R' - w$ is an **RDS** of T, both of which are contradictions. Thus $\deg(w) \geq 3$.

Consider $T' = T - wv$. Let T'_w denote the component of T' containing w and let T'_v denote the component of T' containing v. Since $b_r(T) \geq 2$, it follows that $\gamma_r(T) = \gamma_r(T') = \gamma_r(T'_w) + \gamma_r(T'_v)$. Since v is a remote vertex, it follows that $T'_v \cong S_k$, where $k = |V(T'_v)|$. Therefore, $\gamma_r(T'_v) = k$, and $\gamma_r(T') = \gamma_r(T'_w) + k$. Let R' be a $\gamma_r(T')$-set of T', and notice that $V(T'_v) \subseteq R'$. If $w \notin R'$, then $R = R' - \{v\}$ is a **RDS** of T, a contradiction. Hence, $w \in R'$. Now, since $w \in R'$, every vertex adjacent to w, except possibly x, is in R'. Furthermore, since $\gamma_r(T) = \gamma_r(T')$, R' is a $\gamma_r(T)$-set of T. Since $\mathrm{diam}(T) \geq 4$, it follows that $\deg(x) \geq 2$. If $x \in R'$, then $R = R' - \{w, v\}$ is a **RDS** of T, a contradiction. Thus, $x \notin R'$. Let $N_x = N(x) - \{w\}$ and let $s \in N_x$. If $s \in R'$, then $R = R' - \{w, v\}$ is a **RDS** of T, a contradiction. Hence, $x, s \notin R'$ and x is not a remote vertex. Thus $\deg(s) \geq 2$. Let $N_s = N(s) - \{x\}$. Suppose $N_s \subseteq R'$. Then $R = R' - \{w, v\} \cup \{s\}$ is a **RDS** of T, a contradiction. Thus $N_s \nsubseteq R'$ for all $s \in N_x$. It follows that $R = R' - \{w, v\} \cup \{x\}$ is a **RDS** of T, a contradiction.

Finally, let T be a tree of order $n \geq 4$ such that $b_r(T) = 1$. It follows immediately that $T \ncong S_n$. $\square$

We close this section by determining the restrained bondage numbers for multipartite graphs.

Theorem 7 *Let $n_1 \le n_2 \le \cdots \le n_t$ $(t \ge 2)$, where $n_i \ge 2$ for some $1 \le i \le t$, and let $G = K_{n_1,n_2,\ldots,n_t}$. Then*

$$b_r(G) = \begin{cases} \lceil m/2 \rceil & \text{if } n_m = 1 \text{ and } n_{m+1} \ge 2 \ (1 \le m < t), \\ 2t-2 & \text{if } n_1 = n_2 = \cdots = n_t = 2 \ (t \ge 2), \\ 2 & \text{if } n_1 = 2 \text{ and } n_2 \ge 3 \ (t = 2), \\ \sum_{i=1}^{t-1} n_i - 1 & \text{otherwise.} \end{cases}$$

Proof. Let $1 \le m < t$ such that $n_i = 1$ for $i = 1, \ldots, m$, while $n_i \ge 2$ for $i = m+1, \ldots, t$. Let A be union of the partite sets of cardinality one, and let $B = V(G) - A$.

The following immediate fact will prove to be useful.

Fact. Let H be a graph of order n. If $\Delta(H) \le n-2$, then $\gamma_r(H) \ge 2$.

Let H be a spanning subgraph of G that is obtained by removing fewer than $\lceil \frac{m}{2} \rceil$ edges from G.

If at least one edge incident with every vertex of A is removed to form H, then at least $\lceil \frac{m}{2} \rceil$ edges are removed, which is a contradiction. We conclude that A contains a vertex of degree $n-1$ in H.

If at least $n-2$ edges incident with a vertex of A are removed to form H, then $\lceil \frac{n}{2} \rceil \ge \lceil \frac{m}{2} \rceil > n-2$, i.e. $\lceil \frac{n}{2} \rceil > n-2$, which is a contradiction as $n \ge 4$. We conclude that $\deg_H(v) \ge 2$ for every $v \in A$.

Note that $\deg_G(v) \ge m$ for every $v \in B$, as each vertex of B is adjacent to every vertex of A in G. Thus, for every $v \in B$, $\deg_H(v) \ge m - (\lceil \frac{m}{2} \rceil - 1) = m - \lceil \frac{m}{2} \rceil + 1 = \lfloor \frac{m}{2} \rfloor + 1 \ge 2$.

It now follows that $\gamma_r(H) = 1$, whence $b_r(G) \ge \lceil \frac{m}{2} \rceil$. Furthermore, it follows as in the proof of Proposition 1 that $b_r(G) \le \lceil \frac{m}{2} \rceil$. Hence, $b_r(G) = \lceil \frac{m}{2} \rceil$.

Let $t \ge 2$, assume that $n_1 = n_2 = \cdots = n_t = 2$, and note that $\gamma_r(G) = 2$. If $t = 2$ then $G \cong C_4$, whence $b_r(G) = 2 = 2t-2$. Thus, we assume that $t \ge 3$. We first show that $b_r(G) \ge 2t-2$. Suppose to the contrary that there is a set of edges $E' \subseteq E(G)$ such that $|E'| = 2t-3$ and $\gamma_r(G - E') > \gamma_r(G)$. Notice that $\delta(G - E') \ge 1$. Suppose $u_1 \in V(G - E')$ such that $\deg(u_1) = 1$. Let x be the vertex adjacent to u_1 in $G - E'$ and let $U = \{u_1, u_2\}$ and X be partite sets, with $x \in X$. Moreover, let w be a vertex in a partite set distinct from U and X. Notice that every vertex in $V(G - E') - \{u_1\}$ is adjacent to u_2, and at least one of x or w. It follows that U is a **RDS** of $G - E'$. Hence, $\gamma_r(G - E') \le 2$, a contradiction. Thus, $\delta(G - E') \ge 2$.

We show that this inequality is strict. Suppose $u_1 \in V(G - E')$ such that $\deg(u_1) = 2$ and let $U = \{u_1, u_2\}$ be a partite set. Let $N(u_1) = \{x_1, x_2\}$. Suppose $\{x_1, x_2\}$ is a partite set of $G - E'$. Since $|E'| = 2t-3$, for at least one of x_1, x_2, say x_1, $\deg(x_1) = 2t-2$. Hence, $\{x_1, u_1\}$ is a **RDS** of $G - E'$, a contradiction.

Thus, assume $\{x_1, x_2\}$ is not a partite set of $G - E'$, and let $\{x_1, x_1^*\}$ and $\{x_2, x_2^*\}$ be partite sets.

Notice that U is a dominating set of $G-E'$ except when $u_2x \in E'$ for $x \in Q = V(G-E') - \{x_1, x_2, u_1\}$. Yet, if $x = x_1^*$, then $D = \{u_1, x_2^*\}$ dominates $G-E$, and if $x \in Q - \{x_1^*\}$, then $D = \{u_1, x_1^*\}$ dominates $G-E$. Observe that u_1 is a member of D in each case. Since $2t-4$ edges of E' are incident with u_1 and $\delta(G) = 2t-2$, $\delta((G-E')-D) \geq (2t-2)-3 \geq 1$. Therefore, in each case D is a **RDS** of $G-E'$, a contradiction.

Suppose $u_1 \in V(G-E')$ such that $\deg(u_1) = 2t-2$. Let $U = \{u_1, u_2\}$ be a partite set. Notice that U is a dominating set of $G-E'$. Since $\gamma_r(G-E') > 2$, necessarily U is not restrained. Hence, there exists a vertex $w \in V(G-E')$ such that $N(w) = U$, a contradiction.

Hence, there exists a vertex $x_1 \in V(G-E')$ such that $\deg(x_1) = 2t-3$. Let let $X = \{x_1, x_2\}$ be a partite set, and let y_1 be the one vertex distinct from x_2 that is not adjacent to x_1. Since we assumed that $\gamma_r(G-E') > \gamma_r(G)$, X is not a restrained dominating set of $G-E'$. Since $\delta(G-E') \geq 3$, X is simply not dominating. That is, y_1 is not adjacent to x_2. Let $\{y_1, y_2\}$ be a partite set.

Suppose there is a vertex a which is adjacent to both x_2 and y_1 in $G-E'$. Then $\{x_1, a\}$ is a dominating set of $G-E'$, but as $\gamma_r(G-E') \geq 3$, $\{x_1, a\}$ is not a **RDS** of $G-E'$. Thus, there exists a $b \in V(G-E')$ such that $N_{G-E'}(b) = \{a, x_1\}$, a contradiction. Thus, in $G-E'$, every vertex different from x_1, x_2, y_1, y_2 must be adjacent to at most one of the vertices x_2 and y_1. Since there are $2t-4$ such vertices, each requiring removal of an edge incident with one of x_2 and y_1, we have accounted for at least $2t-4+2 = 2t-2$ edges in E'. Hence, $|E'| \geq 2t-2$, a contradiction.

Thus, $b_r(G) \geq 2t-2$.

Let $\{x_1, x_2\}$ and $\{y_1, y_2\}$ be any partite sets of G and remove all edges incident with x_1, except for x_1y_1. Finally, remove the edge x_2y_2. Let E' be the set of edges removed from G and notice that $|E'| = 2t-2$. Then $3 \leq \gamma_r(G-E')$, whence $b_r(G) = 2t-2$.

Let $t = 2$, and assume $n_1 = 2$ and $n_2 \geq 3$. Notice that $\gamma_r(G) = 2$. Let $X = \{x_1, x_2\}$ and $Y = \{y_1, \ldots, y_{n_2}\}$ be the partite sets of G. Remove any edge e from G. Without loss of generality, suppose $e = x_1y_{n_2}$. Then $G-e$ is simply $K(2, n_2-1)$ with a pendant vertex y_{n_2} attached to x_2. Notice that $\{y_{n_2}, x_1\}$ is a **RDS** of $G-e$. Hence, $\gamma_r(G-e) \leq 2$. Since e was chosen arbitrarily, $b_r(G) \geq 2$. Let E' be the set of edges incident with y_{n_2} and notice that $\gamma_r(G-E') = 3 > \gamma_r(G)$. Thus $b_r(G) \leq |E'| = 2$, and so $b_r(G) = 2$.

Now, assume $n_1 \geq 3$. Notice that $\gamma_r(G) = 2$. Using notation from the previous paragraph, $X = \{x_1, \ldots, x_{n_1}\}$ and $Y = \{y_1, \ldots, y_{n_2}\}$. Let $E' \subset E(G)$ such that $|E'| = n_1 - 2$ and consider $G-E'$. Notice that $\delta(G-E') \geq 2$. Moreover, there is a vertex $x_i \in X$ and a vertex $y_j \in Y$ such that $N_{G-E'}(x_i) = Y$ and $N_{G-E'}(y_j) = X$. Hence, $\gamma_r(G-E') = 2$. Since E' was chosen arbitrarily, $b_r(G) \geq n_1 - 1$. However, notice that $\deg_G(y_1) = n_1$. Let E' be any set of $n_1 - 1$ edges incident with y_1 and notice that $\gamma_r(G-E') = 3 > \gamma_r(G)$. Thus $b_r(G) \leq n_1 - 1$, and so $b_r(G) = n_1 - 1$.

Finally, let $t \geq 3$ and assume $n_1 \geq 2$ and $n_t \geq 3$. Notice that $\gamma_r(G) = 2$. Let $s = \sum_{i=1}^{t-1} n_i$ and observe that $\delta(G) \geq s \geq 4$. We first show that $b_r(G) \geq s-1$. Suppose to the contrary that there exists $E' \subseteq E(G)$ such that $|E'| = s-2$ and $\gamma_r(G-E') > \gamma_r(G)$. Since $\delta(G) \geq s$, it follows that $\delta(G-E') \geq 2$. Suppose there exists $v_1 \in G-E'$ such that $\deg_{G-E'}(v_1)=2$, and let $E(v_1)$ denote the set of edges in G incident with v_1. Since $\deg_{G-E'}(v_1)=2$, it follows that $E' \subset E(v_1)$. Let $\{v_1, v_2\}$ be a partite set and let $y \notin \{v_1, v_2\}$ be a vertex adjacent to v_1 in $G-E'$. Since $\deg_{G-E'}(v_1)=2$, $\{y, v_2\}$ is a **RDS** of $G-E'$. Hence, $\gamma_r(G-E') \leq 2$, which is a contradiction. Therefore, we may assume that $\delta(G-E') \geq 3$.

We claim that each vertex of G is incident with at least one edge in E'. Suppose not. Then there is a vertex $x \in V(G-E')$ such that $\deg_{G-E'}(x) = \deg_G(x)$. Let X be the partite set containing x. Suppose there exists $v \in V(G-E')-X$ such that $X \subseteq N_{G-E'}(v)$. Since $\delta(G-E') \geq 3$, it follows that $\{x, v\}$ is a **RDS** of $G-E'$. Hence, $\gamma_r(G-E') \leq 2$, a contradiction. Thus, for every $v \in V(G-E')-X$, $X \nsubseteq N_{G-E'}(v)$. Since $|V(G-E')-X| \geq s$, it follows that $|E'| \geq s$, contradicting our assumption. Therefore, each vertex of G is incident with at least one edge in E'.

Since $|E'| \leq s-2$, there exists a vertex x_1 that is incident with exactly one edge $e \in E'$. Let $y \in V(G)$ such that $e = yx_1$, and let Y be the partite set containing y. Note that x_1 is adjacent in $G-E'$ to every vertex not in $X \cup \{y\}$. If some vertex $u \notin X \cup Y$ is adjacent to every vertex of $X \cup \{y\}$, then, as $\delta(G-E') \geq 3$, $\{u, x_1\}$ is a **RDS** of $G-E'$, a contradiction. Thus, each vertex not in $X \cup Y$ must be nonadjacent in $G-E'$ to at least one vertex in $X \cup \{y\}$. Moreover, since each vertex of Y is also nonadjacent to some vertex in $G-E'$, it follows that $|E'| \geq |V(G)-(X \cup Y)| + |Y| \geq s$, a contradiction. Therefore, $b_r(G) \geq s-1$.

Finally, let Z be a partite set of G of cardinality n_t, and let $z \in Z$. Notice that $\deg(z) = s$. Let H be the graph obtained by removing $s-1$ edges incident with z. Since $n_t \geq 3$, it follows that $\gamma_r(H) = 3 > \gamma_r(G)$. Thus $b_r(G) \leq s-1$, and so $b_r(G) = s-1$. □

3 Complexity results

Consider the decision problem

RESTRAINED BONDAGE (RB)
INSTANCE: A graph G and a positive integer k.
QUESTION: Does G have a restrained bondage set of cardinality at most k?

Theorem 6 shows that the restrained bondage number of a tree can be computed in constant time. We now show that **RB** is **NP**-complete even for bipartite graphs by describing a polynomial transformation from the following NP-complete problem (see [7]).

3-SATISFIABILITY (3SAT)
INSTANCE: A set $U = \{u_1, u_2, \ldots, u_n\}$ of variables, and a collection $\mathcal{C} = \{C_1, C_2, \ldots, C_m\}$ of clauses over U such that $|C_i| = 3$ for $i = 1, 2, \ldots, m$. Furthermore, every literal is used in at least one clause.
QUESTION: Is there a satisfying truth assignment for $\mathcal{C}$?

Theorem 8 **RB** *is NP-complete, even for bipartite graphs.*

Proof. Let $U = \{u_1, u_2, \ldots, u_n\}$ and $\mathcal{C} = \{C_1, C_2, \ldots, C_m\}$ be an arbitrary instance I of **3SAT**. We will construct a bipartite graph G and an integer k such that I is satisfiable if and only if $b_r(G) \leq k$. The bipartite graph G is constructed as follows. Corresponding to each variable $u_i \in U$, associate a path $P_{u_i} = x_i u_i v_i \overline{u}_i y_i$. Corresponding to each clause $C_j \in \mathcal{C}$, associate a single vertex c_j. Now, join the vertex c_j to a vertex u_i ($\overline{u}_i$, respectively) in P_{u_i} if and only if the literal u_i ($\overline{u}_i$, respectively) appears in clause C_j, for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$. Finally, add a path $P_s = s_1 s_2 s_3$, join s_1 and s_3 to each vertex c_j and set $k = 1$. Throughout, let R be a $\gamma_r(G)$-set.

Claim 1 $\gamma_r(G) \geq 3n+1$. *Moreover, if* $\gamma_r(G) = 3n+1$, *then* $c_j \notin R$ *for each* j, $R \cap V(P_s) = \{s_2\}$, *and* $|R \cap V(P_{u_i})| = 3$ *for each* i.

Proof. Notice that $|R \cap V(P_{u_i})| \geq 3$ for each i, while $|R \cap V(P_s)| \geq 1$. Therefore, $|R| \geq 3n + 1$. Since R was chosen arbitrarily, $\gamma_r(G) \geq 3n + 1$.

Suppose $\gamma_r(G) = 3n + 1$. Then $|R \cap V(P_{u_i})| = 3$ for each i, while $|R \cap V(P_s)| = 1$. Consequently, $c_j \notin R$ for each j. If $s_1 \in R$, then $|R \cap V(P_s)| = 1$ implies that $R \cap V(P_s) = \{s_1\}$, and so s_3 is not dominated. Hence, $s_1 \notin R$, and, similarly, $s_3 \notin R$. Thus, $R \cap V(P_s) = \{s_2\}$. $\diamond$

If $v_i \notin R$ for some i, then $|R \cap \{u_i, \overline{u}_i\}| = 1$; for simplicity denote the neighbor of v_i in R by u_i^*.

Lemma 9 $\gamma_r(G) = 3n + 1$ *if and only if there exists a satisfying truth assignment for* I.

Proof. Suppose $\gamma_r(G) = 3n + 1$. By Claim 1, c_j is adjacent to u_i^* for at least one i. As $|R \cap V(P_{u_i})| = 3$ for each i, it follows that $R \cap V(P_{u_i}) = \{x_i, y_i, u_i^*\}$ or $R \cap V(P_{u_i}) = \{x_i, y_i, v_i\}$.

Now, define $t : U \to \{T, F\}$ by

$$t(u_i) = \begin{cases} T & \text{if } u_i \in R \text{ or } v_i \in R \\ F & \text{if } \overline{u}_i \in R. \end{cases}$$

Consider c_j for some j. Without loss of generality, let c_j be adjacent to $u_i^* \in R$ for some i.

Recall that $u_i^* \in \{u_i, \overline{u}_i\}$. Assume $u_i^* = u_i$. Since u_i is dominating c_j, u_i is in the clause C_j. Since $u_i \in R$, it follows that $t(u_i) = T$. Thus C_j is satisfied. Assume $u_i^* = \overline{u}_i$. Since $\overline{u}_i$ is dominating c_j, $\overline{u}_i$ is in the clause C_j. Since $\overline{u}_i \in R$, it follows that $t(u_i) = F$. Thus, $\overline{u}_i$ is assigned the truth value T, so C_j is satisfied.

Now, let t be a satisfying truth assignment for I. Let R_t be the set of true literals. By construction of G and the fact that t is a satisfying truth assignment for I, each c_j is adjacent to at least one vertex in R_t. Let $R = R_t \cup (\bigcup_{i=1}^{n} \{x_i, y_i\}) \cup \{s_2\}$, and notice that, by definition of R_t, R is a **RDS** of G. Hence, $\gamma_r(G) \leq |R| = 3n + 1$. By Claim 1, $\gamma_r(G) \geq 3n + 1$. Therefore, $\gamma_r(G) = 3n + 1$. $\square$

Lemma 10 *For all* $e \in E(G)$, $\gamma_r(G-e) \leq 3n+2$.

Proof. Since every literal is used in at least one clause, $m \geq 2$. Moreover, each $u_i, \overline{u}_i$ is adjacent to some c_j. Assume $e = s_1s_2$. Then $R' = (\bigcup_{i=1}^{n}\{x_i, y_i, u_i\}) \cup \{s_1, s_2\}$ is a **RDS** of $G-e$. Hence, $\gamma_r(G-e) \leq |R'| = 3n+2$. Similarly, $e = s_2s_3$ implies that $\gamma_r(G-e) \leq 3n+2$.

By the construction of G we need only consider the following cases restricted to a particular vertex c_j. Suppose $e = s_1c_j$. Then $S = (\bigcup_{i=1}^{n}\{x_i, y_i, u_i\}) \cup \{s_1, c_j\}$ is a **RDS** of $G-e$. Hence, $\gamma_r(G-e) \leq |R'| \leq 3n+2$. Similarly, $e = s_3c_j$ implies that $\gamma_r(G-e) \leq 3n+2$. Suppose $e = u_i^*c_j$ for some i. It follows that R' is again a **RDS** of $G-e$. Hence, $\gamma_r(G-e) \leq 3n+2$.

Without loss of generality, assume that c_j is adjacent to u_i, and assume $e = x_iu_i$. Then R' is again a **RDS** of $G-e$. Therefore, $\gamma_r(G-e) \leq |R'| = 3n+2$. Suppose $e = u_iv_i$. Then $R'' = R' - \{u_i\} \cup \{v_i\}$ is a **RDS** of $G-e$ (note that $\deg_{G-e}(\overline{u}_i) \geq 3$, as every literal is contained in some clause). Hence, $\gamma_r(G-e) \leq |R''| \leq 3n+2$. Similar arguments show that $\gamma_r(G-e) \leq 3n+2$ when $e = y_i\overline{u}_i$ or $e = \overline{u}_iv_i$. □

Lemma 11 $\gamma_r(G) = 3n+1$ *if and only if* $b_r(G) = 1$.

Proof. Assume $\gamma_r(G) = 3n+1$. Let $e = s_1s_2$ and consider $G-e$. Suppose $\gamma_r(G) = \gamma_r(G-e)$. Let R' be a $\gamma_r(G-e)$-set of $G-e$. As R' is a $\gamma_r(G)$- set of cardinality $3n+1$, we have (cf. Claim 1) $c_j \notin R'$ for each j and $R' \cap V(P_s) = \{s_2\}$. But then s_1 is not dominated by R', which is a contradiction. Hence, $\gamma_r(G) < \gamma_r(G-e)$, whence $b_r(G) = 1$.

Now, assume $b_r(G) = 1$. By Claim 1, we have that $\gamma_r(G) \geq 3n+1$. Let e' be an edge such that $\gamma_r(G) < \gamma_r(G-e')$. By Lemma 10, we have that $\gamma_r(G-e) \leq 3n+2$ for all $e \in E(G)$. Thus, $3n+1 \leq \gamma_r(G) < \gamma_r(G-e') \leq 3n+2$. It follows that $\gamma_r(G) = 3n+1$. □

Thus, from Lemmas 9 and 11, it follows that $b_r(G) \leq 1$ if and only if I is satisfiable. Hence, we have proven Theorem 8.

4 General bounds and further results

Theorem 12 *If* $\delta(G) \geq 2$, *then* $b_r(G) \leq min\{deg(u) + deg(v) - 2 : uv \in E(G)\}$.

Proof. Let $b_r = \min\{\deg(u) + \deg(v) - 2 : uv \in E(G)\}$, and let $uv \in E(G)$ such that $\deg(u) + \deg(v) - 2 = b_r$. Suppose to the contrary that $b_r(G) > b_r$. Let E' denote the set of edges that are incident with at least one of u and v, but not both. Then $|E'| = b_r$ and $\gamma_r(G - E') = \gamma_r(G)$. Since u and v are endvertices in $G - E'$, it follows that $\gamma_r(G-u-v) = \gamma_r(G) - 2$. Let R be a $\gamma_r(G-u-v)$-set of $G-u-v$. Since $\delta(G) \geq 2$, it follows that $N_G(u) \cup N_G(v) - \{u,v\} \neq \emptyset$. If $N_G(u) \cup N_G(v) - \{u,v\} \subseteq R$, then R is a restrained dominating set of G of cardinality $\gamma_r(G-u-v) = \gamma_r(G) - 2$, a contradiction. Hence, $N(u) \cup N(v) - \{u,v\} \nsubseteq R$ and there is a vertex $w \in N(u) \cup N(v) - \{u,v\}$ such that $w \notin R$. Without loss of generality, assume w is adjacent to u. Then $R \cup \{v\}$ is a restrained dominating set of G of cardinality $\gamma_r(G-u-v) + 1 = \gamma_r(G) - 1$, a contradiction. □

Corollary 13 *If* $\delta(G) \geq 2$*, then* $b_r(G) \leq \Delta(G) + \delta(G) - 2$*.*

Notice that the bounds stated in Theorem 12 and Corollary 13 are sharp. Indeed the class of cycles whose orders are congruent to $1, 2 \bmod 3$ have a restrained bondage number achieving these bounds.

Theorem 14 *If* $\gamma_r(G) \geq 2$*, then* $b_r(G) \leq (\gamma_r(G) - 1)\Delta(G) + 1$*.*

Proof. We proceed by induction on $\gamma_r(G)$. Let $\gamma_r(G) = 2$, and suppose $b_r(G) \geq \Delta(G) + 2$. Let $u \in V(G)$ be of maximum degree. It follows that $\gamma_r(G - u) = \gamma_r(G) - 1 = 1$ and $b_r(G - u) \geq 2$. Since $\gamma_r(G) = 2$ and $\gamma_r(G - u) = 1$, there is a vertex $v \in V(G - u)$ that is adjacent to every vertex in $V(G) - \{u\}$. Furthermore, u is adjacent to every vertex in $V(G) - \{v\}$. Let e be any edge incident with v, and let $H = (V(G - u), E(G - u - e))$. Since $b_r(G - u) \geq 2$, it follows that $\gamma_r(H) = 1$. Hence, there is a vertex $w \in V(G - u)$ such that $w \neq v$ and w is adjacent to every vertex in $V(G - u)$. Since v is the only vertex not in $N_G(u)$, we have $w \in N_G(u)$. Hence, $\deg_G(w) = |V(G)| - 1$, a contradiction. Thus, $b_r(G) \leq \Delta(G) + 1$, for $\gamma_r(G) = 2$.

Now, assume that, for any graph G' such that $\gamma_r(G') = k \geq 2$, $b_r(G') \leq (k - 1)\Delta(G') + 1$. Let G be a graph such that $\gamma_r(G) = k+1$. Suppose to the contrary that $b_r(G) > k\Delta(G)+1$. Let $u \in V(G)$ and notice that $\gamma_r(G - u) = \gamma_r(G) - 1 = k$. Furthermore, $b_r(G) \leq b_r(G - u) + \deg(u)$. By the inductive hypothesis we have $b_r(G) \leq [(k-1)\Delta(G-u)+1] + \deg(u) \leq [(k - 1)\Delta(G - u) + 1] + \Delta(G) = k\Delta(G) + 1$. Thus $b_r(G) \leq k\Delta(G) + 1$, contradicting our assumption that $b_r(G) > k\Delta(G) + 1$. By induction the proof is complete. □

We close by relating the bondage number and restrained bondage number of a graph. Observe that if $\gamma_r(G) = \gamma(G)$, then $b_r(G) \leq b(G)$. Indeed, assume $\gamma_r(G) = \gamma(G)$. Let E' be a set of edges such that $\gamma(G - E') > \gamma(G)$, where $|E'| = b(G)$. Then $\gamma_r(G) = \gamma(G) < \gamma(G - E') \leq \gamma_r(G - E')$, whence $b_r(G) \leq |E'| = b(G)$.

However, we do not have $b_r(G) = b(G)$, when $\gamma_r(G) = \gamma(G)$. Observe that $\gamma_r(K_3) = \gamma(K_3)$, yet $b_r(K_3) = 1$ and $b(K_3) = 2$. We still may not claim that $b_r(G) = b(G)$ even in the case that every $\gamma(G)$-set is a $\gamma_r(G)$-set. The example K_3 again demonstrates this.

Furthermore, we immediately have an infinite class of graphs satisfying $b(G) < b_r(G)$. Define the *brilliant corona* of G to be the graph obtained by attaching $\ell \geq 2$ pendant vertices to each vertex in $V(G)$. The brilliant corona of G will be denoted by $bc(G)$. Let $\mathcal{B} = \{G : G = bc(H)$ for some graph H such that $\delta(H) \geq 2\}$.

Proposition 15 *If* $G \in \mathcal{B}$*, then* $b(G) < b_r(G)$*.*

Proof. Let H be a graph such that $bc(H) = G$, where ℓ is number of pendant vertices attached to each vertex in $V(H)$. Let L denote the set of pendant vertices of G. Notice that L is the unique $\gamma_r(G)$-set of G and $V(H)$ is the unique $\gamma(G)$-set of G. It follows immediately that $b(G) = 1$ and $b_r(G) = \min\{\delta(H), \ell\} \geq 2$. □

Notice that $b_r(G) = \min\{\delta(H), \ell\}$, for $G \in \mathcal{B}$. This fact allows us to show that $b_r(G)$ can be much larger than $b(G)$. We conclude with the following proposition.

Proposition 16 *For each positive integer k there is a graph G such that $k = b_r(G) - b(G)$.*

Proof. Attach no less than $n-1$ pendant vertices to each vertex of K_n and call this new graph G. Let L denote the set of pendant vertices of G. Notice that L is the unique $\gamma_r(G)$-set of G, and $V(K_n)$ is the unique $\gamma(G)$-set of G. It follows immediately that $b(G) = 1$, and $b_r(G) = n - 1$. Thus, $k = b_r(G) - b(G) = n - 2$, and the result follows. □

References

[1] D. Bauer, F. Harary, J. Nieminen and C.L. Suffel, Domination alteration sets in graphs. *Discrete Math.* **47** (1983) 153–161.

[2] G. Chartrand and L. Lesniak, *Graphs & Digraphs: Fourth Edition*, Chapman & Hall, Boca Raton, FL, 2005.

[3] G.S. Domke, J.H. Hattingh, S.T. Hedetniemi, and L.R. Markus, Restrained domination in trees. *Discrete Math.* **211** (2000) 1–9.

[4] G.S. Domke, J.H. Hattingh, M.A. Henning, and L.R. Markus, Restrained domination in graphs with minimum degree two. *J. Combin. Math. Combin. Comput.* **35** (2000) 239–254.

[5] G.S. Domke, J.H. Hattingh, S.T. Hedetniemi, R.C. Laskar, and L.R. Markus, Restrained domination in graphs. *Discrete Math.* **203** (1999), 61–69.

[6] J.F. Fink, M.S. Jacobson, L.F. Kinch, and J. Roberts, The bondage number of a graph. *Discrete Math.* **86** (1990) 47–57.

[7] M.R. Garey and D.S. Johnson, *Computers and Intracability: A Guide to the Theory of NP-Completeness.* Freeman, San Francisco (1979).

[8] B.L. Hartnell, D.F. Rall, Bounds on the bondage number of a graph. *Discrete Math.* **128** (1994), 173–177.

[9] T.W. Haynes, S.T. Hedetniemi, and P.J. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker, New York, 1997.

[10] T.W. Haynes, S.T. Hedetniemi, and P.J. Slater (eds), *Domination in Graphs: Advanced Topics*, Marcel Dekker, New York, 1997.

[11] M.A. Henning, Graphs with large restrained domination number. *Discrete Math.* **197/198** (1999) 415–429.

[12] J.A. Telle and A. Proskurowski, Algoritms for vertex partioning problems on partial k-trees. *SIAM J. Discrete Math.* **10** (1997) 529-550.

[13] U. Teschner, New results about the bondage number of a graph. *Discrete Math.* **171** (1997), 249–259.