On the domination number of Hamiltonian graphs with minimum degree six

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Abstract

Let $G = (V, E)$ be a simple graph. A set $D \subseteq V$ is a *dominating set* of G if every vertex of $V - D$ is adjacent to a vertex of D. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G. We prove that if G is a Hamiltonian graph of order n with minimum degree at least six, then $\gamma(G) \leq \frac{6n}{17}$.

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1 Introduction

In this paper, we follow the notation of [2]. Specifically, let $G = (V, E)$ be a graph with vertex set V and edge set E. A set $S \subseteq V$ is a *dominating set*, denoted **DS**, of G if every vertex not in S is adjacent to a vertex in S. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a **DS**. The concept of domination in graphs, with its many variations, is now well studied in graph theory. The recent book of Chartrand and Lesniak [2] includes a chapter on domination. A thorough study of domination appears in [4, 5].

Ore [7] showed that if G is a graph of order n with $\delta(G) \geq 1$, then $\gamma(G) \leq \frac{n}{2}$, while McCuaig and Shepherd [6] showed that if G is a connected graph of order n with $\delta(G) \geq 2$ and not one of seven exceptional graphs, then $\gamma(G) \leq \frac{2n}{5}$. Moreover, Reed [8] showed that if $\delta(G) \geq 3$, then $\gamma(G) \leq \frac{3n}{8}$. Motivated by these results, Haynes et al. [4] posed the following conjecture.

Conjecture 1 *Let G be a graph of order n such that $\delta(G) \geq k \geq 4$. Then $\gamma(G) \leq \frac{kn}{3k-1}$.*

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The following result is due to Caro and Roditty [1].

Theorem 1 *Let G be a graph of order n. Then*

$$\gamma(G) \leq n[1 - \delta(G)(\frac{1}{\delta(G)+1})^{(1+\frac{1}{\delta(G)})}].$$

We now show that Theorem 1 implies Conjecture 1 for $k \geq 7$.

Proposition 1 *Let G be a graph of order n such that $\delta(G) \geq k \geq 7$. Then*

$$\gamma(G) \leq \frac{kn}{3k-1}.$$

Proof. Suppose $k \geq 7$ and let $\delta(G) \geq k$. We must show that

$$n\Big[1 - \delta(G)\Big(\frac{1}{\delta(G)+1}\Big)^{(1+1/\delta(G))}\Big] \leq \frac{kn}{3k-1}.$$

Thus, it suffices to show that

$$1 - \delta(G)\Big(\frac{1}{\delta(G)+1}\Big)^{(1+1/\delta(G))} \leq \frac{k}{3k-1}.$$

Let $f(x) = 1 - x(\frac{1}{x+1})^{(1+1/x)}$, for $x \geq 7$. Then $f'(x) = -\frac{\ln(x+1)}{x}(\frac{1}{x+1})^{(1+1/x)} < 0$ for $x > 0$. Hence, for $x \geq 7$, f is monotonically decreasing. Since $\delta(G) \geq k \geq 7$, we have $f(\delta(G)) \leq f(k)$; that is, $1 - \delta(G)(\frac{1}{\delta(G)+1})^{(1+1/\delta(G))} \leq 1 - k(\frac{1}{k+1})^{(1+1/k)}$.

Now, let $g(x) = \frac{x}{3x-1}$ and $h(x) = f(x) - g(x)$ for $x \geq 7$. Notice that $g(x) > \frac{1}{3}$, so $h(x) \leq f(x) - \frac{1}{3}$. Let $H(x) = f(x) - \frac{1}{3}$ and notice that, since f is monotonically decreasing, $H(x)$ is also. Then, for $x = 8$, we have $h(8) \leq H(8) = 1 - 8(\frac{1}{9})^{(1+1/8)} - \frac{1}{3} < 0$. Since h is monotonically decreasing, it follows that $h(x) \leq h(8) \leq 0$ for $x \geq 8$. Moreover, for $x = 7$, we have $h(7) = 1 - 7(\frac{1}{8})^{(1+1/7)} - \frac{7}{3(7)-1} < 0$. Thus, $1 - k(\frac{1}{k+1})^{(1+1/k)} \leq \frac{k}{3k-1}$, for $k \geq 7$, and the result follows. □

Sohn and Yuan [9] proved that Conjecture 1 holds for graphs with minimum degree four, while Xing et al. [10] proved that Conjecture 1 holds for graphs with minimum degree five. Therefore, Conjecture 1 remains open for graphs with minimum degree six. In the next section we prove that Conjecture 1 holds for Hamiltonian graphs with minimum degree six.

2 Main result

We first provide some definitions and notation. Let C be a cycle and P be a path with $V(C) \cap V(P) = \emptyset$. Let $v \in V(C)$ and let x be an endvertex of P. Let $V' = V(C) \cup V(P)$

and let $E' = E(C) \cup E(P) \cup \{vx\}$. We call the graph $L = (V', E')$ a *lasso*. The cycle C is called the *body* of L. If L is a subgraph of a graph G, then L is called a *lasso* of G. In order to prove that Conjecture 1 holds for Hamiltonian graphs with minimum degree six, we state several preliminary results.

Lemma 1 *For $k \geq 1$, let $P = x_1, x_2, \cdots, x_{3k+1}$ be a path of order $3k+1$. If x_1 is adjacent to a vertex x_{3i} for some $1 \leq i \leq k$, then P can be dominated by k vertices.*

Proof. The set $D = \{x_3, x_6, \cdots, x_{3k}\}$ is a **DS** set of P such that $|D| = k$. $\diamond$

Lemma 2 *For $k \geq 1$, let C be a cycle of order $3k+1$, and $P = x_1, x_2, x_3$ be a path such that $V(C) \cap V(P) = \emptyset$. If x_2 has a neighbor on C, then $C \cup P$ can be dominated by $k+1$ vertices.*

Proof. Let $C = y_1, y_2, \cdots, y_{3k+1}, y_1$ and, without loss of generality, assume x_2 is adjacent to y_1. Then $D = \{x_2, y_3, y_6, \cdots, y_{3k}\}$ is a **DS** of $C \cup P$ such that $|D| = k+1$. $\diamond$

The following result is due to Clark and Dunning [3].

Lemma 3 *Let G be a graph of order n with $\delta(G) \geq 4$. If $n \leq 16$, then $\gamma(G) \leq \frac{n}{3}$.*

The following result is due to Xing et al. [10].

Lemma 4 *Let G be a graph of order $3k+1$, where $2 \leq k \leq 8$. If $\delta(G) \geq 5$, then $\gamma(G) \leq k$.*

We are now in position to prove our main result.

Theorem 2 *Let G be a Hamiltonian graph of order n such that $\delta(G) \geq 6$. Then*

$$\gamma(G) \leq \frac{6n}{17}.$$

Proof. Let $V(G) = \{1, 2, \cdots, n\}$ and, without loss of generality, assume $C = 1, 2, \cdots, n, 1$ is a Hamiltonian cycle of G. If $n \leq 16$, then, by Lemma 3, $\gamma(G) \leq \frac{n}{3} \leq \frac{6n}{17}$. Thus, $n \geq 17$. Now, let $k \geq 6$ and consider the following cases.

Case 1. $n = 3k - 1$.

Then $D = \{2, 5, \ldots, 3k-1\}$ is a **DS** set of G such that $|D| = k = \frac{n+1}{3}$. Since $n \geq 17$, it follows that $\gamma(G) \leq \frac{n+1}{3} \leq \frac{6n}{17}$.

Case 2. $n = 3k$.

Then $D = \{2, 5, \cdots, 3k-1\}$ is a **DS** of G such that $|D| = k = \frac{n}{3}$. It follows that $\gamma(G) \leq \frac{n}{3} \leq \frac{6n}{17}$.

Case 3. $n = 3k + 1$.

If $k \leq 8$, then $n \leq 25$, and by Lemma 4, $\gamma(G) \leq k \leq \frac{6n}{17}$. Suppose $k \geq 11$. Then $n \geq 34$ and $D = \{2, 5, \cdots, 3k-1, 3k+1\}$ is a **DS** of G such that $|D| = k+1 = \frac{n+2}{3}$. Since $n \geq 34$, it follows that $\gamma(G) \leq \frac{n+2}{3} \leq \frac{6n}{17}$. Hence, we only need to verify that if G has order $n = 28$ ($n = 31$, respectively), then G has a **DS** of cardinality 9 (10, respectively).

Since the proofs are similar, we consider only $n = 31$. The proof is by contradiction, that is, we assume $\gamma(G) \geq 11$. Since $\delta(G) \geq 6$, each vertex of G is incident with at least four chords of C. We choose a lasso L of G of order 31, obtainable from C, such that the number of vertices comprising the body of L is maximum. That is, L is a spanning subgraph of the union of C and a chord of C.

Let $v \in V(G)$. Suppose, without loss of generality, that $1v$ is a chord of C such that $1, v, v-1, \cdots, 1$ is the body of L. Note that 1 is adjacent to both v and 31. We consider possible values of v. If 1 is adjacent to $3i$ for some $1 \leq i \leq 10$, then, by Lemma 1, G can be dominated by 10 vertices, which is a contradiction. Thus we may assume that 1 is not adjacent to $3i$ for all i. By similar reasoning, 31 is not adjacent to $3i-1$ for all i. Since the body of L is a maximum, and by re-labeling if necessary, we have that $v \geq 17$. Since 1 is not adjacent to $3i$ for all i, we have $v \in \{17, 19, 20, 22, 23, 25, 26, 28, 29\}$.

Before proceeding further, we bound the adjacencies of vertices 31 and 30. Suppose b (c, respectively) is adjacent to 31 (30, respectively). Then we obtain lassos L_1 and L_2 (L_1' and L_2', respectively) with cycle lengths $b+1$ and $32-b$ ($c+2$ and $31-c$, respectively). Thus, $b+1 \leq v$ and $32-b \leq v$ ($c+2 \leq v$ and $31-c \leq v$, respectively), and so $32-v \leq b \leq v-1$ ($31-v \leq c \leq v-2$, respectively).

Case 3.1. $v = 17$.

Then 31 is possibly adjacent to vertices in $\{32-17, \ldots, 17-1, 1, 30\} = \{15, 16, 1, 30\}$, contradicting the fact that $\deg(v) \geq 6$.

Case 3.2. $v \in \{19, 22, 25, 28\}$.

Since $31-v \leq c \leq v-2$, 30 is adjacent to some vertex on the cycle $1, v, v-1, \cdots, 2, 1$. As $v \equiv 1 \bmod 3$, Lemma 2 implies that G can be dominated by 10 vertices, a contradiction.

Case 3.3. $v = 20$.

Again we check the possible adjacencies of 31. By reasoning similar to Case 3.1, we have that 31 is adjacent to 1, 30 and possibly $12, 13, \cdots, 19$. Recall that 31 is not adjacent to $3i-1$ for all $1 \leq i \leq 10$. Thus 31 is not adjacent to 14 or 17. Since $\deg(31) \geq 6$, 31 must be adjacent to at least one of the vertices 12, 15 or 18. Then $D = \{3, 6, 9, 12, 15, 18, 20, 23, 26, 29\}$ is a **DS** of G of cardinality 10, a contradiction.

Case 3.4. $v = 23$.

Initially, 31 is adjacent to 1, 30, and possibly vertices in $\{9, 10, \cdots, 21, 22\}$. Let $D = \{3, 6, 9, 12, 15, 18, 21, 23, 26, 29\}$. Then D dominates G if 31 is adjacent to $3i$ for some $1 \leq i \leq 7$. Hence, we eliminate these possibilities and also vertices of the form $3i-1$. We now have that 31 is possibly adjacent to vertices in $\{10, 13, 16, 19, 22\}$. Since $\deg(31) \geq 6$, 31 must be adjacent to either 19 or 22.

Now consider the adjacencies of 30. Initially, 30 is adjacent to 29, 31 and possibly vertices in $\{8, 9, \cdots, 20, 21\}$. Let $D' = \{1, 3, 6, 9, 12, 15, 18, 21, 25, 28\}$. Then D' dominates G if 30 is adjacent to $3i$ for some $1 \leq i \leq 7$. Hence, we eliminate these possibilities and also the vertices of the form $3i - 2$. We now have that 30 is possibly adjacent to the vertices in $\{8, 11, 14, 17, 20\}$. Since $\deg(30) \geq 6$, 30 must be adjacent to either 8 or 11. Then $D'' = \{2, 5, 8, 11, 14, 17, 19, 22, 25, 28\}$ is a **DS** of G of cardinality 10, a contradiction.

Case 3.5. $v = 26$.

Initially, 31 is adjacent to 1, 30, and possibly vertices in $\{6, 7, \cdots, 25\}$. Let $D = \{3, 6, 9, 12, 15, 18, 21, 24, 26, 29\}$. Then D dominates G if 31 is adjacent to $3i$ for some $1 \leq i \leq 8$. Hence, we eliminate these possibilities and also vertices of the form $3i - 1$. Thus, we have that 31 is possibly adjacent to the vertices in $\{7, 10, 13, 16, 19, 22, 25\}$. Since $\deg(31) \geq 6$, 31 must be adjacent to at least one of the vertices in $\{16, 19, 22, 25\}$.

Now consider the adjacencies of 30. Initially, 30 is adjacent to 29, 31 and possibly vertices in $\{5, 6, \cdots, 24\}$. Let $D' = \{1, 3, 6, 9, 12, 15, 18, 21, 24, 28\}$. Then D' dominates G if 30 is adjacent to $3i$ for some $1 \leq i \leq 8$. Hence, we eliminate these possibilities and also the vertices of the form $3i - 2$. We now have that 30 is possibly adjacent to vertices in $\{5, 8, 11, 14, 17, 20, 23\}$. Since $\deg(30) \geq 6$, 30 must be adjacent to at least one of the vertices in $\{5, 8, 11, 14\}$. Then $D'' = \{2, 5, 8, 11, 14, 16, 19, 22, 25, 28\}$ is a **DS** of G of cardinality 10, a contradiction.

Case 3.6. $v = 29$.

Initially, 31 is adjacent to 1, 30, and possibly the vertices in $\{3, 4, \cdots, 28\}$. Let $D = \{3, 6, 9, 12, 15, 18, 21, 24, 27, 29\}$. Then D dominates G if 31 is adjacent to $3i$ for some $1 \leq i \leq 9$. Hence, we eliminate these possibilities and also vertices of the form $3i - 1$. Thus, we have that 31 is possibly adjacent to the vertices in $\{4, 7, 10, 13, 16, 19, 22, 25, 28\}$.

Now, consider the adjacencies of 30. Initially, 30 is adjacent to 29, 31 and possibly the vertices in $\{2, 3, \cdots, 27\}$. Let $D' = \{1, 3, 6, 9, 12, 15, 18, 21, 25, 28\}$. Then D' dominates G if 30 is adjacent to $3i$ for some $1 \leq i \leq 9$. Hence, we eliminate these possibilities and also the vertices of the form $3i - 2$. We now have that 30 is possibly adjacent to the vertices in $\{2, 5, 8, 11, 14, 17, 20, 23, 26\}$.

Suppose 31 is adjacent to one of the vertices in $\{19, 22, 25, 28\}$. Let $D'' = \{2, 5, 8, 11, 14, 17, 19, 22, 25, 28\}$. Then D'' dominates G if 30 is adjacent to $3i - 1$ for some $1 \leq i \leq 6$. Hence, we eliminate these possibilities. It follows that 30 is adjacent to 29, 31 and possibly to vertices in $\{20, 23, 26\}$, which implies that $\deg(30) \leq 5$, a contradiction. We conclude that 31 is not adjacent to any of the vertices in $\{19, 22, 25, 28\}$.

Suppose 31 is adjacent to 4. Then 2 must be adjacent to some vertex on the cycle $31, 4, 5, \cdots, 30, 31$ of length 28. By Lemma 2, G can be dominated by 10 vertices, which is a contradiction. Suppose 31 is adjacent to 7. Then 2 must be adjacent to some vertex on the cycle $31, 7, 8, \cdots, 30, 31$ of length 25. By Lemma 2, the vertices on the cycle and the vertices 1, 2, 3 can be dominated by a set composed of 9 vertices. Adding the vertex 5 to this set yields a **DS** set of G of cardinality 10, a contradiction. Thus, 31 is adjacent to 1, 30 and possibly to vertices in $\{10, 13, 16\}$, which implies that $\deg(31) \leq 5$, a contradiction. □

Corollary 1 *Let G be a Hamiltonian graph of order n such that $\delta(G) \geq k \geq 3$. Then $\gamma(G) \leq \frac{kn}{3k-1}$.*

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